The Scott Adjunction

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Thm. (Rosicky, Beke, Lieberman)

A category $\mathcal{A}$ is equivalent to an abstract elementary class iff:

1. it is an accessible category with directed colimits;
2. every map is a monomorphism;
3. it has a *structural* functor $U : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is finitely accessible and U is iso-full, nearly full and preserves directed colimits and monomorphisms.

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Quite not what we were looking for, uh?!

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Our aim

1. Have a **conceptual understanding** of those accessible categories in which model theory blooms naturally.
2. When an accessible category with directed colimits admits such a nice forgetful functor?

The Scott Adjunction (Henry, DL)

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1. Acc_{ω} is the 2-category of accessible categories with directed colimits, a 1-cell is a functor preserving directed colimits, 2-cells are invertible natural transformations.
2. Topoi is the 2-category of Groethendieck topoi. A 1-cell is a geometric morphism and has the direction of the right adjoint. 2-cells are natural transformation between left adjoints.

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There is an accessible category with directed colimits which cannot be axiomatized by a geometric theory.

This problem was originally proposed by Rosicky in his talk "Towards categorical model theory" at the 2014 category theory conference in Cambridge: *Show that the category of uncountable sets and monomorphisms between cannot be obtained as the category of point of a topos. Or give an example of an abstract elementary class that does not arise as the category points of a topos.*

The Scott construction

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Let $\mathcal{A}$ be a 0-cell in Acc_ω. $\mathsf{S}(\mathcal{A})$ is defined as the category $\mathsf{Acc}_\omega(\mathcal{A}, \mathsf{Set})$.Let $f : \mathcal{A} \to \mathcal{B}$ be a 1-cell in Acc_ω.

$$
\begin{tikzcd}
\mathcal{A} \arrow[d, "f"] & & \mathsf{S}\mathcal{A} \arrow[d, bend left, "f_*"] \arrow[d, phantom, "f^* \dashv f_*"] \\
\mathcal{B} & & \mathsf{S}\mathcal{B} \arrow[u, bend left, "f^*"]
\end{tikzcd}
$$

$\mathsf{S}f = (f^* \dashv f_*)$ is defined as follows: f^* is the precomposition functor $f^*(g) = g \circ f$. This is well defined because f preserve directed colimits. f^* preserve all colimits and thus has a right adjoint, that we indicate with f_*. Observe that f^* preserve finite limits because finite limits commute with directed colimits in Set.

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In this perspective our adjunction, which in this case is a duality, presents $\mathsf{S}(\mathcal{A})$ as a free geometric theory attached to the accessible category $\mathcal{A}$ that is willing to axiomatize $\mathcal{A}$.

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Is the Scott adjunction the categorification of the Isbell duality between locales and topological spaces?

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Is the Scott adjunction the categorification of the Isbell duality between locales and topological spaces?

Not precisely.

The geometric picture

$$\begin{tikzcd}
 & \mathsf{Loc} \arrow[dl, bend left, "\mathsf{pt}"] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathsf{Top} \arrow[ur, bend left, "\mathcal{O}"] & & \mathsf{Pos}_\omega \arrow[ll, "\mathsf{ST}"] \arrow[ul, bend left, "\mathsf{S}"]
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Loc is the category of Locales. It is defined to be the opposite category of frames, where objects are frames and morphisms are morphisms of frames.

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& \mathsf{Topoi} \arrow[dl, bend left=15, "pt"] \arrow[dr, bend left=15, "\mathsf{pt}"] & \\
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Ionads!

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The 2-category of Ionads was introduced by Garner. A **ionad** $\mathcal{X} = (X, \mathrm{Int})$ is a set X together with a comonad $\mathrm{Int} : \mathrm{Set}^X \to \mathrm{Set}^X$ preserving finite limits. While topoi are the categorification of locales, Ionads are the categorification of the notion of topological space, to be more precise, Int categorifies the interior operator of a topological space.

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Thm. (Garner)

The category of coalgebras for a ionad is indicated with $\mathbb{O}(\mathcal{X})$ and is a cocomplete elementary topos. A ionad is bounded if $\mathbb{O}(\mathcal{X})$ is a Grothendieck topos. Thus one should look at the functor

$$\mathbb{O} : \mathsf{BIon} \to \mathsf{Topoi},$$

as the categorification of the functor that associates to a space its frame of open sets.

$$
\begin{tikzcd}
& \mathsf{Topoi} \arrow[dl, dashed, bend left] \arrow[dr, bend left, "\mathsf{pt}"] & \\
\mathsf{BIon} \arrow[ur, bend left, "\mathbb{O}"] & & \mathsf{Acc}_\omega \arrow[ll, dashed] \arrow[ul, bend left, "\mathsf{S}"]
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 & \mathsf{Topoi} \arrow[dl, dashed, bend left] \arrow[dr, bend left, "\mathsf{pt}"] & \\
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\end{tikzcd}$$

Unfortunately the definition of Garner does not allow to find a right adjoint for $\mathbb{O}$.

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\end{tikzcd}
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Unfortunately the definition of Garner does not allow to find a right adjoint for $\mathbb{O}$.
In order to fix this problem, one needs to stretch Garner definition and introduce **large (bounded) Ionads**.

Thm. (DL)

Replacing bounded Ionads with large bounded Ionads, there exists a right adjoint for $\mathbb{O}$ and a Scott topology-construction ST such that $\mathsf{S} = \mathbb{O} \circ \mathsf{ST}$, in complete analogy to the posetal case.

$$
\begin{tikzcd}
 & \mathsf{Topoi} \arrow[dl, bend left=15, "pt"] \arrow[dr, bend left=15, "\mathsf{pt}"] & \\
\mathsf{LBIon} \arrow[ur, bend left=15, "\mathbb{O}"] & & \mathsf{Acc}_\omega \arrow[ll, "\mathsf{ST}"] \arrow[ul, bend left=15, "\mathsf{S}"]
\end{tikzcd}
$$