

# The Scott Adjunction

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A category  $\mathcal{A}$  is equivalent to an abstract elementary class iff:

- 1 it is an accessible category with directed colimits;
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*Quite not what we were looking for, uh?!*





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- 1 Have a **conceptual understanding** of those accessible categories in which model theory blooms naturally.
- 2 When an accessible category with directed colimits admits such a nice forgetful functor?



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- 2  $\text{Topoi}$  is the 2-category of Groethendieck topoi. A 1-cell is a geometric morphism and has the direction of the right adjoint. 2-cells are natural transformation between left adjoints.





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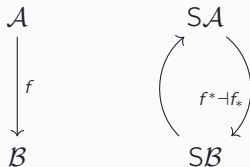
This problem was originally proposed by Rosicky in his talk “Towards categorical model theory” at the 2014 category theory conference in Cambridge: *Show that the category of uncountable sets and monomorphisms between cannot be obtained as the category of point of a topos. Or give an example of an abstract elementary class that does not arise as the category points of a topos.*

## The Scott construction

Let  $\mathcal{A}$  be a 0-cell in  $\text{Acc}_\omega$ .  $S(\mathcal{A})$  is defined as the category  $\text{Acc}_\omega(\mathcal{A}, \text{Set})$ .

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Let  $\mathcal{A}$  be a 0-cell in  $\text{Acc}_\omega$ .  $S(\mathcal{A})$  is defined as the category  $\text{Acc}_\omega(\mathcal{A}, \text{Set})$ . Let  $f : \mathcal{A} \rightarrow \mathcal{B}$  be a 1-cell in  $\text{Acc}_\omega$ .



$Sf = (f^* \dashv f_*)$  is defined as follows:  $f^*$  is the precomposition functor  $f^*(g) = g \circ f$ . This is well defined because  $f$  preserve directed colimits.  $f^*$  preserve all colimits and thus has a right adjoint, that we indicate with  $f_*$ . Observe that  $f^*$  preserve finite limits because finite limits commute with directed colimits in  $\text{Set}$ .

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In this perspective our adjunction, which in this case is a duality, presents  $S(\mathcal{A})$  as a free geometric theory attached to the accessible category  $\mathcal{A}$  that is willing to axiomatize  $\mathcal{A}$ .

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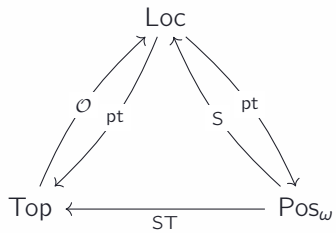
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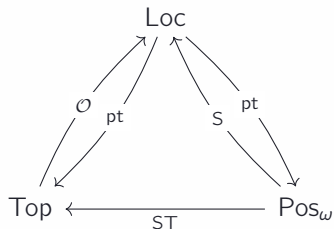
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*Not precisely.*

## The geometric picture



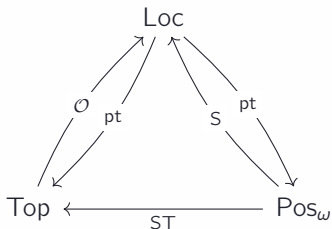
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$\text{Loc}$  is the category of Locales. It is defined to be the opposite category of frames, where objects are frames and morphisms are morphisms of frames.



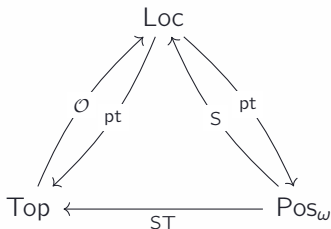
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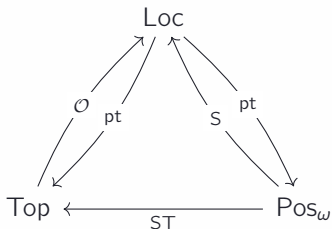


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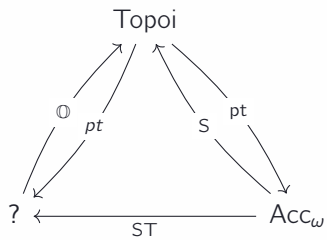
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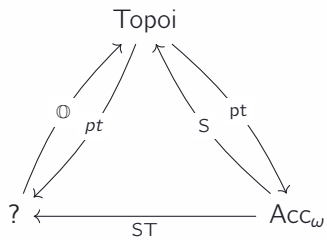


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lonads!



## Ionads

The 2-category of ionads was introduced by Garner. A **ionad**  $\mathcal{X} = (X, \text{Int})$  is a set  $X$  together with a comonad  $\text{Int} : \text{Set}^X \rightarrow \text{Set}^X$  preserving finite limits. While topoi are the categorification of locales, ionads are the categorification of the notion of topological space, to be more precise,  $\text{Int}$  categorifies the interior operator of a topological space.

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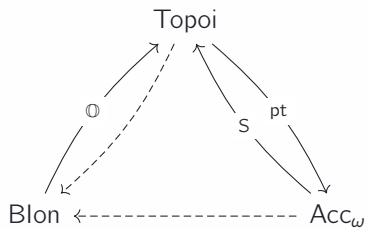
The category of coalgebras for a ionad is indicated with  $\mathbb{O}(\mathcal{X})$  and is a cocomplete elementary topos. A ionad is bounded if  $\mathbb{O}(\mathcal{X})$  is a Grothendieck topos. Thus one should look at the functor

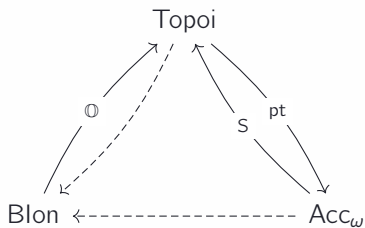
$$\mathbb{O} : \text{Blon} \rightarrow \text{Topoi},$$

as the categorification of the functor that associates to a space its frame of open sets.

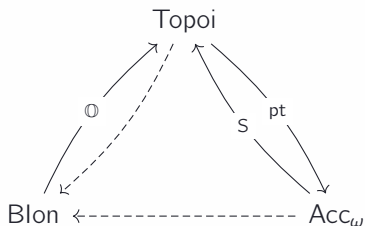








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In order to fix this problem, one needs to stretch Garner definition and introduce **large (bounded) lonads**.



### Thm. (DL)

Replacing bounded Ionads with large bounded Ionads, there exists a right adjoint for  $\mathbb{O}$  and a Scott topology-construction  $ST$  such that  $S = \mathbb{O} \circ ST$ , in complete analogy to the posetal case.

