Set-theoretic remarks on a possible definition of elementary ∞-topos

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Geometric ∞-toposes

Definition

An ∞-category $\mathcal{X}$ is called a geometric ∞-topos if there is a small ∞-category $\mathcal{C}$ and an adjunction

$$\begin{tikzcd} \mathcal{P}(\mathcal{C}) \arrow[r, shift left=2, "L"] \arrow[r, phantom, "\perp"] & \mathcal{X} \arrow[l, hook', shift left=2, "i"] \end{tikzcd}$$

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In particular, every geometric ∞-topos is presentable.

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Dependent sums always exist by universal property of pullbacks.

Proposition

In a geometric ∞-topos all dependent products exist.

Classifiers

Let S be a class of morphisms in an ∞-category $\mathcal{E}$, which is closed under pullbacks.
A classifier for the class S is a morphism $t : \bar{U} \to U$ such that for every object X the operation of pulling back defines an equivalence of ∞-groupoids

$$\mathrm{Map}(X, U) \simeq (\mathcal{E}^S_{/X})^{\sim}$$

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Theorem (Rezk)

In a geometric ∞-topos, there are arbitrarily large cardinals κ such that the class S_κ of relatively κ-compact morphisms has a classifier.

Elementary ∞-toposes

Definition (Shulman)

An elementary ∞-topos is an ∞-category $\mathcal{E}$ such that

1. *$\mathcal{E}$ has all finite limits and colimits.*
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We will only focus on a subaxiom of (4):

Definition

We say that a class of morphisms S satisfies (DepProd) if it has a classifier and it is closed under dependent products

Uniformization

We will need a fundamental result:

Theorem (Adámek, Rosický for the 1-dimensional case)

Given a small family $(f_i : \mathcal{K}_i \to \mathcal{L}_i)_{i \in I}$ of accessible functors between presentable ∞-categories, there are arbitrarily large cardinals κ such that all functors f_i's preserve κ-compact objects.

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- *We may assume that κ-compact objects in a presheaf ∞-category are precisely the objectwise κ-compact presheaves.*
- *Given a diagram shape R, we may assume that κ-compact objects are stable under R-limits.*
- *We may assume that many such properties hold for the same cardinal.*

Main result

Theorem

Fixing a Grothendieck universe $\mathcal{U}$, every geometric ∞-topos satisfies (DepProd) if and only if there are unboundedly many inaccessible cardinals below the cardinality of $\mathcal{U}$.

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First, prove $\Leftarrow$.

Step 1. In the ∞-category $\mathcal{S}$ of spaces, if κ is inaccessible then κ-compact objects are stable under exponentiation.

Main result

Step 2. In $\mathcal{P}(\mathcal{C})$, given presheaves F and G, their exponential F^G is given by the formula

$$F^G(C) = \int_{D \in \mathcal{C}} \mathrm{Map}(\mathrm{Map}(D, C) \times G(D), F(D))$$

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$\Rightarrow$ κ-compact presheaves are stable under exponentiation.

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Step 3. Given an adjunction

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making $\mathcal{X}$ a geometric ∞-topos, choose κ such that (Step 2) holds in $\mathcal{P}(\mathcal{C})$.

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The properties of $L \dashv i$ will transfer stability of κ-compact objects under exponentiation to $\mathcal{X}$.

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Step 4. Given an object $p : Z \to X$ in $\mathcal{X}_{/X}$, its dependent product along a terminal morphism $X \to *$ is given by

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Step 5. For generic dependent products, decompose the codomain as a colimit of compact objects Y_i's and then choose κ such that (Step 4) holds in all ∞-toposes $\mathcal{X}_{/Y_i}$.

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For a discrete space X, the terminal morphism $X \to *$ is contained in a class S having a classifier $t : \bar{U} \to U$ such that

$$\exists \quad \begin{tikzcd} Y \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\ulcorner", very near start] & \bar{U} \arrow[d, "t"] \\ Z \arrow[r] & U \end{tikzcd} \quad , \qquad \begin{tikzcd} Z \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & \bar{U} \arrow[d, "t"] \\ W \arrow[r] & U \end{tikzcd}$$

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- Assume that all fibers of t are discrete.
- For each point in U, its fiber along t can be regarded as a set.

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- $\kappa > |X|$.
- For $\lambda, \mu < \kappa$, closure under dependent products $\Rightarrow \mu^\lambda < \kappa$.
- In non-trivial cases, $\sum_{i \in I} \alpha_i \leq \prod_{i \in I} \alpha_i \Rightarrow \kappa$ is regular.

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Given a geometric ∞-topos $\mathcal{X}$, take

$$\mathcal{X}^{\mu} \subset \mathcal{X}.$$

$\Rightarrow \mathcal{X}^{\mu}$ is not a geometric ∞-topos (it doesn't have all small colimits), but it is an elementary ∞-topos.

Thank you!