Set-theoretic remarks on a possible definition of elementary ∞ -topos

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11 July 2019

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Definition

An ∞ -category \mathcal{X} is called a geometric ∞ -topos if there is a small ∞ -category \mathcal{C} and an adjunction

$$\mathcal{P}(\mathcal{C}) \xrightarrow[i]{} \stackrel{L}{\underset{i}{\longleftarrow}} \mathcal{X}$$

where i is full and faithful, $L \circ i$ is accessible and L preserves all finite limits.

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In particular, every geometric ∞ -topos is presentable.

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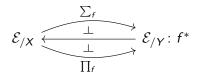
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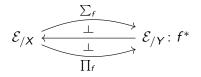
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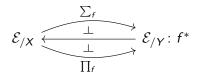


Remark

Dependent sums always exist by universal property of pullbacks.

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Remark

Dependent sums always exist by universal property of pullbacks.

Proposition

In a geometric ∞ -topos all dependent products exist.

Let S be a class of morphisms in an ∞ -category \mathcal{E} , which is closed under pullbacks.

A classifier for the class S is a morphism $t : \overline{U} \to U$ such that for every object X the operation of pulling back defines an equivalence of ∞ -groupoids

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Theorem (Rezk)

In a geometric ∞ -topos, there are arbitrarily large cardinals κ such that the class S_{κ} of relatively κ -compact morphisms has a classifier.

Elementary ∞ -toposes

Definition (Shulman)

An elementary ∞ -topos is an ∞ -category $\mathcal E$ such that

- \mathcal{E} has all finite limits and colimits.
- **2** \mathcal{E} is locally Cartesian closed.
- Solution The class of all monomorphisms in $\mathcal E$ admits a classifier.

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- Solution For each morphism f in E there is a class of morphisms S ∋ f such that S has a classifier and is closed under limits and colimits taken in overcategories and under dependent sums and products.

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- The class of all monomorphisms in E admits a classifier.
- So For each morphism f in E there is a class of morphisms S ∋ f such that S has a classifier and is closed under limits and colimits taken in overcategories and under dependent sums and products.

We will only focus on a subaxiom of (4):

Definition

We say that a class of morphisms S satisfies (DepProd) if it has a classifier and it is closed under dependent products

Uniformization

We will need a fundamental result:

Theorem (Adámek, Rosický for the 1-dimensional case)

Given a small family $(f_i : \mathcal{K}_i \to \mathcal{L}_i)_{i \in I}$ of accessible functors between presentable ∞ -categories, there are arbitrarily large cardinals κ such that all functors f_i 's preserve κ -compact objects.

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Example

 We may assume that κ-compact objects in a presheaf ∞-category are precisely the objectwise κ-compact presheaves. We will need a fundamental result:

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- We may assume that many such properties hold for the same cardinal.

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Theorem

Fixing a Grothendieck universe \mathcal{U} , every geometric ∞ -topos satisfies (DepProd) if and only if there are unboundedly many inaccessible cardinals below the cardinality of \mathcal{U} .

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First, prove \Leftarrow .

Step 1. In the ∞ -category S of spaces, if κ is inaccessible then κ -compact objects are stable under exponentiation.

Step 2. In $\mathcal{P}(\mathcal{C})$, given presheaves F and G, their exponential F^G is given by the formula

$${\sf F}^{\sf G}({\sf C}) = \int_{D\in {\cal C}} {\sf Map}({\sf Map}(D,{\sf C}) imes {\sf G}(D),{\sf F}(D))$$

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By uniformization, we may choose a cardinal κ such that:

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- κ-compact spaces are stable under exponentiation (Step 1)
- κ -compact spaces are stable under C-ends
- $\Rightarrow \kappa$ -compact presheaves are stable under exponentiation.

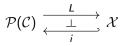
Step 3. Given an adjunction

$$\mathcal{P}(\mathcal{C}) \xrightarrow[i]{} \stackrel{L}{\underset{i}{\longleftarrow}} \mathcal{X}$$

making \mathcal{X} a geometric ∞ -topos, choose κ such that (Step 2) holds in $\mathcal{P}(\mathcal{C})$.

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The properties of $L \dashv i$ will transfer stability of κ -compact objects under exponentiation to \mathcal{X} .

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Step 4. Given an object $p: Z \to X$ in $\mathcal{X}_{/X}$, its dependent product along a terminal morphism $X \to *$ is given by

$$\prod_{X} p = Z^X \times_{X^X} \{p\}$$

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Step 5. For generic dependent products, decompose the codomain as a colimit of compact objects Y_i 's and then choose κ such that (Step 4) holds in all ∞ -toposes $\mathcal{X}_{/Y_i}$.

Now prove \Rightarrow .

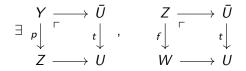
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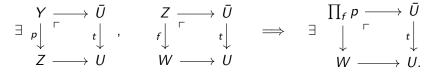
For a discrete space X, the terminal morphism $X \to *$ is contained in a class S having a classifier $t : \overline{U} \to U$ such that



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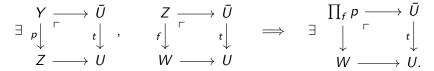
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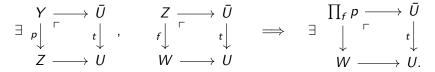


• Assume that all fibers of *t* are discrete.

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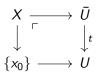


• Assume that all fibers of *t* are discrete.

• For each point in U, its fiber along t can be regarded as a set.

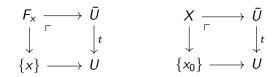
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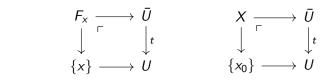
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- In non-trivial cases, $\sum_{i \in I} \alpha_i \leq \prod_{i \in I} \alpha_i \Rightarrow \kappa$ is regular.

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Given a geometric ∞ -topos \mathcal{X} , take

 $\mathcal{X}^{\mu} \subset \mathcal{X}.$

 $\Rightarrow \mathcal{X}^{\mu}$ is not a geometric ∞ -topos (it doesn't have all small colimits), but it is an elementary ∞ -topos.

Thank you!

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