Involutive factorisation systems & Dold-Kan correspondences

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[1]joint with Christophe Cazanave and Ingo Waschkies

Theorem (Dold 1958, Kan 1958)

$$M : \underline{\mathrm{Ab}}^{\Delta^{\mathrm{op}}} \simeq \mathrm{Ch}(\mathbb{Z}) : K$$

Corollary

There is a simplicial abelian group $K(A,n)$ such that $\pi_n(K(A,n)) = A$ and $\pi_i(K(A,n)) = 0$ for $i \neq n$.

Proof.

$K : \mathrm{Ch}(\mathbb{Z}) \to \underline{\mathrm{Ab}}^{\Delta^{\mathrm{op}}}$ takes homology into homotopy. $K(A,n)$ is the image of the chain complex: $0 \leftarrow \cdots \leftarrow 0 \leftarrow \overset{n}{A} \leftarrow 0 \leftarrow \cdots$ $\square$

Purpose of the talk

Categorical structure of Δ inducing Dold-Kan correspondence.

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Definition (simplex category Δ)

$\mathrm{Ob}\Delta = \{[n] = \{0, 1 \ldots, n\}, n \geq 0\}$, $\mathrm{Mor}\Delta = \{\text{monotone maps}\}$

Remark (epi-mono factorisation system)

The category Δ is generated by elementary

- *face operators* $\epsilon_i^n : [n-1] \to [n]$, $0 \leq i \leq n$, and
- *degeneracy operators* $\eta_i^n : [n+1] \to [n]$, $0 \leq i \leq n$.

Every simplicial operator $\phi : [m] \to [n]$ factors as

$$\begin{tikzcd}
{[m]} \arrow[rr, "\phi"] \arrow[dr, "epi"'] & & {[n]} \\
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and every epi (resp. mono)morphism in Δ is a canonical composite of elementary degeneracy (resp. face) operators.

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Definition (geometric realisation, Milnor 1957)

$\Delta \hookrightarrow \mathrm{Top} : [n] \mapsto \Delta_n$ yields by left Kan extension along Yoneda

$$|-|_\Delta : \mathrm{Sets}^{\Delta^{\mathrm{op}}} \to \mathrm{Top}.$$

Theorem (Quillen 1968)

Geometric realisation is left part of a Quillen equivalence.

Definition (simplicial homology, Eilenberg 1944)

$$\begin{array}{ccccccc} \mathrm{Sets}^{\Delta^{\mathrm{op}}} & \longrightarrow & \underline{\mathrm{Ab}}^{\Delta^{\mathrm{op}}} & \xrightarrow{N} & \mathrm{Ch}(\mathbb{Z}) & \longrightarrow & \underline{\mathrm{Ab}}^{\mathbb{N}} \\ X_\bullet & \longmapsto & \mathbb{Z}[X_\bullet] & \longmapsto & (N_\bullet(X), d_\bullet) & \longmapsto & H_\bullet(X) \end{array}$$

where $(N_n(X) = \mathbb{Z}[X_n]/\mathbb{Z}[D_n(X)], d_n = \sum_k (-1)^k X(\epsilon_k^n))$

is isomorphic to the

Moore chain complex $(M_n(X) = \bigcap_{0 \leq k < n} X(\epsilon_k^n), d_n = X(\epsilon_n^n))$.

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Proposition (Dold 1958)

Moore normalisation M admits a left adjoint K assigning to a chain complex $(C_\bullet, d_\bullet)$ the simplicial abelian group

$$K(C_\bullet, d_\bullet)_n = \bigoplus_{[n]\twoheadrightarrow[k]} C_k \text{ with } K(\phi) : \bigoplus_{[n]\twoheadrightarrow[k]} C_k \to \bigoplus_{[m]\twoheadrightarrow[j]} C_j$$

$$\text{where } K(\phi)_{ab} = \begin{cases} d_k \text{ if } \begin{tikzcd} {[m]} \arrow[r, "\phi"] \arrow[d, "a"'] & {[n]} \arrow[d, "b"] \\ {[k-1]} \arrow[r, tail, "\epsilon_k^k"'] & {[k]} \end{tikzcd} \\ 0 \text{ otherwise} \end{cases}$$

Remark

- *unit:* $\forall C_\bullet \in \mathrm{Ch}(\mathbb{Z})$ one has $C_\bullet \cong MKC_\bullet$ $\rightsquigarrow$ easy
- *counit:* $\forall A_\bullet \in \underline{\mathrm{Ab}}^{\Delta^{op}}$ one has $KMA_\bullet \cong A_\bullet$ $\rightsquigarrow$ difficult

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Definition (Involutive factorisation system)

A factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ is called *involutive* if there is a specified faithful, identity-on-objects functor $(-)^* : \mathcal{E}^{\mathrm{op}} \to \mathcal{M}$ sth.

(I1) $ee^* = 1$ (the split idempotent e^*e is called an *$\mathcal{E}$-projector*);

(I2) the morphisms f^*e form a subcategory of $\mathcal{C}$;

(I3) $\forall (A \xrightarrow{m} B) \in \mathcal{M}\ \forall \phi \in \mathrm{Proj}_{\mathcal{E}}(A)\ \exists \psi \in \mathrm{Proj}_{\mathcal{E}}(B) : m\phi = \psi m$;

(I4) $\mathrm{Proj}_{\mathcal{E}}(A)$ is finite. *Primitive* $\mathcal{E}$-projectors can be linearly ordered such that if ϕ precedes ψ then $\psi\phi$ is an $\mathcal{E}$-projector.

Remark (primitive $\mathcal{E}$-projectors)

$\mathrm{Proj}_{\mathcal{E}}(A) \cong \mathrm{Quot}_{\mathcal{E}}(A)$. Primitive $\mathcal{E}$-projectors are *covered* by 1_A.

Remark (Involutive factorisation system for Δ)

Each epi $e : [m] \twoheadrightarrow [n]$ has a *maximal* section $e^* : [n] \to [m]$.
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For any small category $\mathcal{A}$ the category $\Delta \wr \mathcal{A}$ is defined by

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Definition (Joyal 1997, B 2007)

Put $\Theta_1 = \Delta$ and for $n > 1 : \Theta_n = \Delta \wr \Theta_{n-1}$

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- $(\phi; \phi_{ij}) : ([m], A_1, \ldots, A_m) \to ([n], B_1, \ldots, B_n))$ is given by $\phi : [m] \to [n]$ and $A_i \to B_j$ whenever $\phi(i-1) < j \leq \phi(i)$

Definition (Joyal 1997, B 2007)

Put $\Theta_1 = \Delta$ and for $n > 1 : \Theta_n = \Delta \wr \Theta_{n-1}$

Theorem (Makkai-Zawadowski 2003, B 2003)

Θ_n embeds densely into nCat, i.e. there is a fully faithful functor

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Definition (elegant Reedy category=skeletal EZ-category)

A *Reedy category* $\mathcal{C}$ has a strict $(\mathcal{E},\mathcal{M})$-factorisation system, a grading $\deg : \mathrm{Ob}\mathcal{C} \to \mathbb{N}$ such that $\mathcal{E}$ (resp. $\mathcal{M}$)-maps lower (resp. increase) degree. $\mathcal{C}$ is *elegant* if $\mathcal{E}$ has absolute pushouts.

Lemma (generalised Eilenberg-Zilber)

For any presheaf $X : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$, each $x \in X(c)$ equals $X(\phi)(y)$ for unique $\phi : c \to d$ in $\mathcal{E}$ and "non-degenerate" $y \in X(d)$.

Proposition (Bergner-Rezk 2017)

If $\mathcal{A}$ is an elegant Reedy category then so is $\Delta \wr \mathcal{A}$.
In particular, Θ_n is an elegant Reedy category.

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Theorem (BCW 2019)

$$\underline{\mathrm{Ab}}^{\Theta_n^{\mathrm{op}}} \simeq [\Xi_{\Theta_n}^{\mathrm{op}}, \underline{\mathrm{Ab}}]_*$$

Remark (Θ_n-set model for Eilenberg-MacLane spaces)

- For each abelian group A there is an abelian group object B^nA in nCat with one k-cell for $0 \leq k < n$;
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Example (cells of $K(\mathbb{Z}/2\mathbb{Z}, n)$ for $n = 1, 2, 3$)

# cells in dim	0	1	2	3	4	5	6	7	8	9
$K(\mathbb{Z}/2\mathbb{Z}, 1)$	1	1	1	1	1	1	1	1	1	1
$K(\mathbb{Z}/2\mathbb{Z}, 2)$	1	0	1	1	2	3	5	8	13	21
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