Simplicial objects and relative monotone-light factorization in Mal'tsev categories

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Categorical Galois Theory

Framework that allows the study of extensions or coverings of objects of a category. Examples include

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1. Galois theory of commutative rings
2. Central extensions of groups, or more generally exact Mal'tsev categories
3. Coverings of locally connected spaces

Categorical Galois Theory

Definition (Galois structure)

A Galois structure $\Gamma = (\mathcal{A}, \mathcal{X}, I, U, \mathcal{F})$ consists of a category $\mathcal{A}$ together with a full reflective subcategory $\mathcal{X}$ and a class $\mathcal{F}$ of fibrations containing isomorphisms and stable under pullbacks, composition and preserved by the reflector I.

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This data induces an adjunction
$$\begin{tikzcd} \mathcal{A} \downarrow_{\mathcal{F}} B \arrow[r, bend left, "I^B"] \arrow[r, phantom, "\perp"] & \mathcal{X} \downarrow_{\mathcal{F}} IB. \arrow[l, bend left, "H^B = \eta_B^*"] \end{tikzcd}$$

Admissibility

We will be interested in cases where H^B is fully faithful.
An object B with this property is called *admissible*, and a Galois structure is admissible if every object of $\mathcal{A}$ is admissible.

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An object B with this property is called *admissible*, and a Galois structure is admissible if every object of $\mathcal{A}$ is admissible.
This is equivalent to the reflector I preserving the pullbacks of the form

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U(X) \arrow[d, "U(f)"] \\ Z \arrow[r] & U(Y) \end{tikzcd}$$

where X, Y are in $\mathcal{X}$ and $f \in \mathcal{F}$.

Trivial coverings

A fibration $f : A \to B$ lies in the essential image of the right adjoint H^B iff the square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & UI(A) \arrow[d, "UI(f)"] \\ B \arrow[r, "\eta_B"'] & UI(B) \end{tikzcd}$$

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is a pullback.

These fibrations are called *trivial coverings*.

Coverings

Every fibration $h : X \to Y$ induces a pair of adjoint functors $h_! \dashv h^*$:

- the pullback $h^* : \mathcal{A} \downarrow_{\mathcal{F}} Y \to \mathcal{A} \downarrow_{\mathcal{F}} X$;
- the composition $h_! : \mathcal{A} \downarrow_{\mathcal{F}} X \to \mathcal{A} \downarrow_{\mathcal{F}} Y$.

h is an effective $\mathcal{F}$-descent morphism if h^* is monadic.

Example

If $\mathcal{C}$ is exact and $\mathcal{F} = \{\text{regular epis}\}$, then every $h \in \mathcal{F}$ is an effective $\mathcal{F}$-descent morphism.

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A fibration f is called a *covering* if it is a locally trivial covering, i.e. if $h^*(f)$ is a trivial covering for some effective $\mathcal{F}$-descent morphism h.

Example : Groupoids and simplicial sets

Theorem (Gabriel, Zisman [8])

The nerve functor $N : \mathbf{Grpd} \to \mathbf{Simp}$ *is fully faithful, and has a left adjoint* π_1*, the fundamental groupoid. Thus* **Grpd** *can be identified with a reflective subcategory of* **Simp**.

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Theorem (Brown, Janelidze [1])

If $\mathcal{F}$ is the class of all Kan fibrations, then every Kan simplicial object is admissible. The coverings are "second order coverings", characterized by a certain unique lifting property.

Mal'tsev categories

A finitely complete category is a Mal'tsev category if every reflexive relation is an equivalence relation.

Proposition (Carboni, Lambek, Pedicchio, 1991 [3])

If $\mathcal{C}$ is a regular category, the following are equivalent:

- *$\mathcal{C}$ is Mal'tsev.*
- *$R \circ S = S \circ R$ for any internal equivalence relations R, S.*
- *$R \circ S$ is an equivalence relation for any equivalence relations R, S.*

If $\mathcal{C}$ is a variety, then this is equivalent to the existence of a ternary operation p satisfying $p(x, y, y) = x$ and $p(y, y, z) = z$.

Examples : **Grp** ($p(x, y, z) = xy^{-1}z$), R-**Alg**, **Lie**, any additive category, **Grp**(**Top**), the dual of any topos...

Example : Birkhoff subcategories of Mal'tsev categories

Definition (Birkhoff subcategory)

A Birkhoff subcategory of a regular category $\mathcal{C}$ is a full reflective subcategory closed under quotients and subobjects.

Example

For varieties of universal algebras, Birkhoff subcategories coincide with subvarieties.

Theorem (Janelidze, Kelly [10])

Every Birkhoff subcategory $\mathcal{X}$ of an exact Mal'tsev category $\mathcal{A}$ gives an admissible Galois structure $(\mathcal{A}, \mathcal{X}, I, U, \mathcal{F})$ where $\mathcal{F}$ is the class of regular epimorphisms.

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Example (Gran [9])

Any category $\mathcal{C}$ can be identified with the category of discrete internal groupoids. $\pi_0 : \mathbf{Grpd}(\mathcal{C}) \to \mathcal{C}$ makes it a reflective, and in fact Birkhoff, subcategory. The coverings are precisely the regular epimorphic discrete fibrations.

Theorem (Carboni, Kelly, Pedicchio [2]/Everaert, Goedecke, Van der Linden [7, 6])

A regular category $\mathcal{C}$ is Mal'tsev if and only if every simplicial object in $\mathcal{C}$ satisfies the Kan property.
In that case every regular epimorphism in $\mathbf{Simp}(\mathcal{C})$ is a Kan fibration.

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This raises the question : is the inclusion $\mathbf{Grpd}(\mathcal{C}) \to \mathbf{Simp}(\mathcal{C})$ part of an admissible Galois structure when $\mathcal{C}$ is exact Mal'tsev ?

A simplicial object is a groupoid if and only if every square

$$\begin{tikzcd} X_{n+2} \arrow[r, "d_j"] \arrow[d, "d_i"'] & X_{n+1} \arrow[d, "d_i"] \\ X_{n+1} \arrow[r, "d_{j-1}"'] & X_n \end{tikzcd}$$

is a pullback.
When $\mathcal{C}$ is regular Mal'tsev, these are all regular pushouts : the canonical map $X_{n+2} \to X_{n+1} \times_{X_n} X_{n+1}$ is always a regular epi.

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When $\mathcal{C}$ is regular Mal'tsev, these are all regular pushouts : the canonical map $X_{n+2} \to X_{n+1} \times_{X_n} X_{n+1}$ is always a regular epi.
Thus $\mathbb{X}$ is a groupoid if and only if these maps are all monomorphisms.

We denote D_i the kernel pair of $d_i : X_n \to X_{n-1}$. For $n \geq 2$, we define

$$H_n(\mathbb{X}) = \bigvee_{0 \leq i < j \leq n} D_i \wedge D_j.$$

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For $n \geq 2$, $d_i(H_{n+1}(\mathbb{X})) = H_n(\mathbb{X})$, thus the d_i induce maps $\frac{X_{n+1}}{H_{n+1}(\mathbb{X})} \to \frac{X_n}{H_n(\mathbb{X})}$.

In order to factor $d_1 : X_2 \to X_1$ through the quotient $X_2 \to \frac{X_2}{H_2(\mathbb{X})}$, we would need to check that $D_0 \wedge D_2 \leq D_1$, or equivalently $d_1(D_0 \wedge D_2) = \Delta$. So we define $H_1(\mathbb{X}) = d_1(D_0 \wedge D_2)$.

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So we define $H_1(\mathbb{X}) = d_1(D_0 \wedge D_2)$.
In fact

$$d_1(D_0 \wedge D_2) = d_0(D_1 \wedge D_2) = d_2(D_0 \wedge D_1).$$

Theorem (D.)

Let $\mathcal{C}$ be an exact Mal'tsev category and $\mathbb{X} \in \mathbf{Simp}(\mathcal{C})$, and let us define $\overline{X_n} = X_n / H_n(\mathbb{X})$. Then

- $\overline{\mathbb{X}}$ *can be endowed with the structure of a simplicial object;*
- $\overline{\mathbb{X}}$ *is a groupoid;*
- *any morphism $f : \mathbb{X} \to \mathbb{Y}$ where $\mathbb{Y}$ is a groupoid factorizes through $\overline{\mathbb{X}}$.*

Corollary (D.)

$\mathbf{Grpd}(\mathcal{C})$ *is a Birkhoff subcategory of* $\mathbf{Simp}(\mathcal{C})$*; in particular, if $\mathcal{F}$ is the class of regular epimorphisms in* $\mathbf{Simp}(\mathcal{C})$*, then* $\Gamma = (\mathbf{Simp}(\mathcal{C}), \mathbf{Grpd}(\mathcal{C}), I, U\mathcal{F})$ *is an admissible Galois structure.*

A fibration $f : \mathbb{X} \to \mathbb{Y}$ is a trivial covering if and only if $F_n \wedge H_n(\mathbb{X}) = \Delta_{X_n}$ for all $n \geq 1$.

Theorem (D.)

A fibration is a covering if and only if

$$d_1(F_2 \wedge D_0 \wedge D_2) = \Delta_{X_1}$$

and

$$\bigvee_{0 \leq i < j \leq n} (F_n \wedge D_i \wedge D_j) = \Delta_{X_n}$$

for all $n \geq 2$.

Factorizations

Let $\Gamma = (\mathcal{A}, \mathcal{X}, I, U, \mathcal{F})$ be an admissible Galois structure, with $\mathcal{F}$ the class of all morphisms.
Then any arrow f has a reflection in the subcategory of trivial coverings, given by

$$\begin{tikzcd}
A \arrow[dr, "e"] \arrow[drr, bend left, "\eta_A"] \arrow[ddr, bend right, "f"'] & & \\
& B \times_{I(B)} I(A) \arrow[r] \arrow[d, "m"] & I(A) \arrow[d, "I(f)"] \\
& B \arrow[r, "\eta_B"] & I(B).
\end{tikzcd}$$

Then $I(e)$ is an isomorphism.
This gives a *factorization system* $(\mathcal{E}, \mathcal{M})$.

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To make the factorization system stable, we must

- stabilize $\mathcal{E}$, by replacing it with the class $\mathcal{E}'$ of morphisms stably in $\mathcal{E}$.
- localize $\mathcal{M}$, by replacing it with the class $\mathcal{M}^*$ of coverings.

The resulting classes are still orthogonal, but it is not always true that every fibration has a factorization.

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The resulting classes are still orthogonal, but it is not always true that every fibration has a factorization.
When this happens, we say that Γ has an associated monotone-light factorization system $(\mathcal{E}', \mathcal{M}^*)$.

Relative factorization systems

But what if $\mathcal{F}$ is not the class of all morphisms?

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Not every morphism has a factorization, but every fibration does.
Moreover the orthogonality is preserved, and $\mathcal{M} \subset \mathcal{F}$.
This is a *relative* factorization system for $\mathcal{F}$ in the sense of Chikhladze [4].

Stabilization/localization can be generalized the relative case, to give a stable *relative* factorization system $(\mathcal{E}', \mathcal{M}^*)$ where

- $\mathcal{E}'$ is the class of morphisms where every pullback *along a morphism in* $\mathcal{F}$ is in $\mathcal{E}$;
- $\mathcal{M}^*$ is again the class of locally trivial covering.

Proposition (Carboni, Janelidze, Kelly, Paré / Chikhladze)

If for every B there exists an $\mathcal{F}$-effective descent morphism $p : E \to B$ where E has the property that the factorization of every $g : C \to E$ in $\mathcal{F}$ is stable under pullbacks along maps in $\mathcal{F}$, then $(\mathcal{E}', \mathcal{M}^*)$ is a relative factorization system.

Such an object E is called a *stabilizing object*.

Example (Chikhladze [4])

The Galois structure of Brown and Janelidze, given by the nerve functor between groupoids and Kan complexes, admits a relative monotone-light factorization system for Kan fibrations.

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Example (Cigoli, Everaert, Gran [5])

When $\mathcal{C}$ is exact Mal'tsev, the Galois structure $(\mathbf{Grpd}(\mathcal{C}), \mathcal{C}, \pi_0, D, \mathcal{F})$ admits a relative monotone-light factorization system for regular epimorphisms.

Coverings are discrete fibrations and $\mathcal{E}'$ is the class of final functors ; so this monotone-light relative factorization system is the restriction of the comprehensive factorization system to regular epimorphism.

Both proofs rely on showing that $Dec(\mathbb{X})$ is a stabilizing object.

Definition

A simplicial object $\mathbb{X}$ is called exact if the canonical maps $\kappa_n : X_n \to K_n(\mathbb{X})$ to the simplicial kernels of $\mathbb{X}$ are all regular epimorphisms.

Example

For a simplicial object $\mathbb{X} = (X_n)_{n \geq 0}$, let $Dec(\mathbb{X})$ be the simplicial object $(X_{n+1})_{n \geq 0}$, with the same face and degeneracies except the $d_{n+2} : X_{n+2} \to X_{n+1}$ and $s_{n+1} : X_{n+1} \to X_{n+2}$. Then $Dec(\mathbb{X})$ is an exact simplicial object.

Theorem (D.)

In an exact Mal'tsev category $\mathcal{C}$*, every exact simplicial object is a stabilizing object. In particular, since for every object* $\mathbb{X}$ *we have a regular epimorphism* $Dec(\mathbb{X}) \to \mathbb{X}$ *defined by the* $d_{n+1} : X_{n+1} \to X_n$*, the Galois structure* $\Gamma = (\mathbf{Simp}(\mathcal{C}), \mathbf{Grpd}(\mathcal{C}), I, U, \mathcal{F})$ *admits a relative monotone-light factorization system.*

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