Weak adjoint functor theorems

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joint work with John Bourke and Lukáš Vokřínek

Adjoint Functor Theorems

ur-AFT

category $\mathcal{B}$ with all limits
$U \colon \mathcal{B} \to \mathcal{A}$ preserves them

U has left adjoint

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General AFT

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Solution Set Condition

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more Adjoint Functor Theorems

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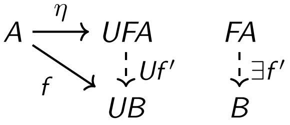

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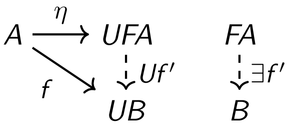

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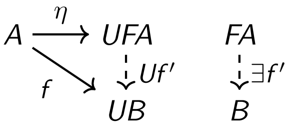
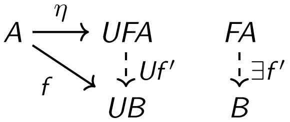


Enriched weakness



$$\mathcal{B}(FA, B) \xrightarrow{\text{surj.}} \mathcal{A}(A, UB)$$

- enriched categories have homs $\mathcal{C}(C, D)$ lying in $\mathcal{V}$
- (Lack-Rosicky) “Enriched Weakness” uses class $\mathcal{E}$ of morphisms in $\mathcal{V}$ to play the role of surjections
- $\mathcal{V} = \textbf{Set}$, $\mathcal{E} = \{\text{surjections}\}$ gives unenriched weakness
- $\mathcal{E} = \{\text{isomorphisms}\}$ gives “non-weak weakness”

Enriched weakness

$$\begin{array}{ccc} A & \xrightarrow{\eta} & UFA \\ & \searrow^{f} & \Big\downarrow{\scriptstyle Uf'} \\ & & UB \end{array} \qquad \begin{array}{c} FA \\ \Big\downarrow{\scriptstyle \exists f'} \\ B \end{array} \qquad\qquad \mathcal{B}(FA, B) \xrightarrow{\mathcal{E}\text{-map}} \mathcal{A}(A, UB)$$

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Examples

$$\mathcal{B}(FA, B) \xrightarrow{\mathcal{E}} \mathcal{A}(A, UB)$$

$\mathcal{V}$	$\mathcal{E}$	$\mathcal{E}$-weak left adjoint
Set	isos	left adjoint
Set	surjections	weak left adjoint
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Cat	surj equivalences	(...) left biadjoint
sSet	dual strong deformation retracts	

Definition

A morphism $p: X \to Y$ of simplicial sets is *dsdr* if it is contractible in **sSet**$/Y$:

- it has a section s
- with a homotopy $s \circ p \sim 1_X$
- such that induced homotopy $p \circ s \circ p \sim p$ is trivial.

The setting

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An *interval* in $\mathcal{V}$ is a factorization

$$I + I \xrightarrow{i} J \xrightarrow{w} I, \qquad w \circ i = \nabla$$

of the codiagonal with $i \in \mathcal{I}$ and $w \in \mathcal{W}$.

- A morphism $A \to B$ in a $\mathcal{V}$-category $\mathcal{C}$ is a morphism $I \to \mathcal{C}(A, B)$.
- A homotopy between morphisms $A \to B$ is a morphism $J \to \mathcal{C}(A, B)$ (for some interval)
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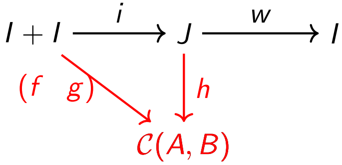

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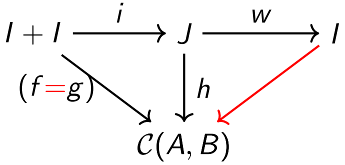

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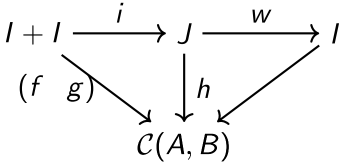

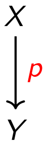

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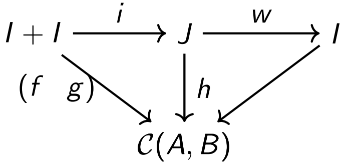

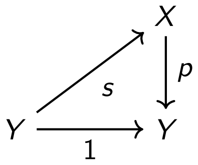

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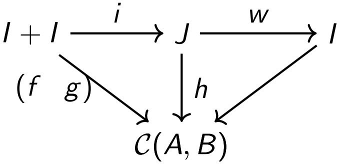

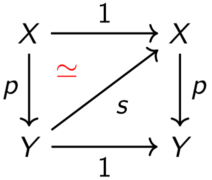

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sSet	mono	wk hty equiv	(Quillen) dsdr
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The limits in question

Limit of a functor $S\colon \mathcal{D} \to \mathcal{B}$ is defined by natural isos

$$\mathcal{B}(B, \lim S) \cong [\mathcal{D}, \mathbf{Set}](\Delta 1, \mathcal{B}(B, S)).$$

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A weight $G\colon \mathcal{D} \to \mathcal{V}$ is cofibrant if it is projective with respect to the pointwise trivial fibrations:

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$$\begin{array}{ccc} & H & HD \\ & \downarrow p & \downarrow pD \in \mathcal{P} \\ G & K & KD \end{array}$$

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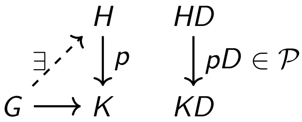

A $\mathcal{V}$-category $\mathcal{B}$ has enough cofibrant limits if for any weight G there is a cofibrant G' with a pointwise trivial fibration $G' \to G$ for which $\mathcal{B}$ has G'-weighted limits.

The VGAFT

Let $\mathcal{V}$ be a monoidal model category with cofibrant unit.

V(ery) G(eneral) AFT

$\mathcal{V}$-category $\mathcal{B}$ with all powers and enough cofibrant limits
$U \colon \mathcal{B} \to \mathcal{A}$ preserves them
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N(ot) Q(uite) S(o) G(eneral) AFT

$\mathcal{V}$-category $\mathcal{B}$ with all powers and enough cofibrant limits
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$\mathcal{B}_0$ and $\mathcal{A}_0$ are accessible, U_0 accessible functor (unenriched)

U has a dsdr-weak left adjoint

Application (to Riehl-Verity ∞-cosmoi)

An ∞-cosmos is a **sSet**-category with all powers, enough cofibrant limits, and certain further structure.

These are intended to be a model-independent framework in which to study the totality of $(\infty, 1)$-categories and related structures.

Corollary

Any accessible ∞-cosmos has dsdr-weak colimits.

A cosmological functor is an enriched functor between ∞-cosmoi which preserves this structure.

Corollary

Any cosomological functor satisfying the SSC has a dsdr-weak left adjoint.

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