Enriched Regular Theories

Giacomo Tendas

Joint work with:
Stephen Lack

8 July 2019

MACQUARIE
University

Outline

Theories in Logic

A theory is given by a list of axioms on a fixed set of operations; its models are corresponding sets and functions that satisfy those axioms.

Examples

1. Algebraic Theories: axioms consist of equations based on the operation symbols of the language;
2. Essentially Algebraic Theories: axioms are still equations but the operation symbols are not defined globally, but only on equationally defined subsets;
3. Regular Theories: we allow existential quantification over the usual equations.

Theories in Category Theory

Categorically speaking, we could think of a theory as a category $\mathcal{C}$ with some structure, and of a model of $\mathcal{C}$ as a functor $F : \mathcal{C} \to \mathbf{Set}$ which preserves that structure.

Examples

1. Algebraic Theories: categories with finite products; their models are finite product preserving functors [Lawvere,63].
2. Essentially Algebraic Theories: categories with finite limits; lex functors are its models [Freyd,72].
3. Regular Theories: regular categories; their models are regular functors [Makkai-Reyes,77].

Gabriel-Ulmer Duality

- The two notions of theory, categorical and logical, can be recovered from each other: given a logical theory, produce a category with the relevant structure for which models of the theory correspond to functors to **Set** preserving this structure, and vice versa.

For essentially algebraic theories there is a duality between theories and their models:

Theorem (Gabriel-Ulmer)

The following is a biequivalence of 2-categories:

$$\mathrm{Lfp}(-, \mathbf{Set}) : \mathbf{Lfp} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathbf{Set}).$$

Gabriel-Ulmer Duality

- The two notions of theory, categorical and logical, can be recovered from each other: given a logical theory, produce a category with the relevant structure for which models of the theory correspond to functors to **Set** preserving this structure, and vice versa.

For essentially algebraic theories there is a duality between theories and their models:

Theorem (Gabriel-Ulmer)

The following is a biequivalence of 2-categories:

$$\mathrm{Lfp}(-, \mathbf{Set}) : \mathbf{Lfp} \rightleftarrows \mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathbf{Set}).$$

Regular and Exact Categories

Regular Categories: finitely complete ones with coequalizers of kernel pairs, for which regular epimorphisms are pullback stable.

Theorem (Barr's Embedding)

Let $\mathcal{C}$ be a small regular category; then the evaluation functor $\mathrm{ev} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathbf{Set}), \mathbf{Set}]$ *is fully faithful and regular.*

Exact Categories: regular ones with effective equivalence relations.

Theorem (Makkai's Image Theorem)

Let $\mathcal{C}$ be a small exact category. The essential image of the embedding $\mathrm{ev} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathbf{Set}), \mathbf{Set}]$ *is given by those functors which preserve filtered colimits and small products.*

Regular and Exact Categories

Regular Categories: finitely complete ones with coequalizers of kernel pairs, for which regular epimorphisms are pullback stable.

Theorem (Barr's Embedding)

Let $\mathcal{C}$ be a small regular category; then the evaluation functor $\mathrm{ev} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathbf{Set}), \mathbf{Set}]$ *is fully faithful and regular.*

Exact Categories: regular ones with effective equivalence relations.

Theorem (Makkai's Image Theorem)

Let $\mathcal{C}$ be a small exact category. The essential image of the embedding $\mathrm{ev} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathbf{Set}), \mathbf{Set}]$ *is given by those functors which preserve filtered colimits and small products.*

Duality for Exact Categories

- On one side of the duality there is the 2-category **Ex** of exact categories, regular functors, and natural transformations.
- On the other side is a 2-category **Def** whose objects are called definable categories and correspond to models of regular theories.

Theorem (Prest-Rajani/Kuber-Rosický)

The following is a biequivalence of 2-categories:

$$\mathrm{Def}(-, \mathbf{Set}) : \mathbf{Def} \rightleftarrows \mathbf{Ex}^{op} : \mathrm{Reg}(-, \mathbf{Set})$$

Base for Enrichment

Let $\mathcal{V} = (\mathcal{V}_0, I, \otimes)$ be a symmetric monoidal closed category.

Recall: An object A of $\mathcal{V}_0$ is called finitely presentable if the hom-functor $\mathcal{V}_0(A, -) : \mathcal{V}_0 \to \mathbf{Set}$ preserves filtered colimits; denote by $(\mathcal{V}_0)_f$ the full subcategory of finitely presentable objects.

Definition (Kelly)

We say that $\mathcal{V} = (\mathcal{V}_0, I, \otimes)$ is a locally finitely presentable as a closed category if:

1. $\mathcal{V}_0$ is cocomplete with strong generator $\mathcal{G} \subseteq (\mathcal{V}_0)_f$ (i.e. is locally finitely presentable) ;
2. $I \in (\mathcal{V}_0)_f$;
3. if $A, B \in \mathcal{G}$ then $A \otimes B \in (\mathcal{V}_0)_f$.

Duality

- An object A of $\mathcal{L}$ is called finitely presentable if the hom-functor $\mathcal{L}(A, -) : \mathcal{L} \to \mathcal{V}$ preserves conical filtered colimits;
- Locally finitely presentable $\mathcal{V}$-category: $\mathcal{V}$-cocomplete with a small strong generator consisting of finitely presentable objects;
- Finitely complete $\mathcal{V}$-category: one with finite conical limits and finite powers.

Theorem (Kelly)

The following is a biequivalence of 2-categories:

$$(-)_f^{op} : \mathcal{V}\text{-}\mathbf{Lfp} \rightleftarrows \mathcal{V}\text{-}\mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathcal{V})$$

Duality

- An object A of $\mathcal{L}$ is called finitely presentable if the hom-functor $\mathcal{L}(A, -) : \mathcal{L} \to \mathcal{V}$ preserves conical filtered colimits;
- Locally finitely presentable $\mathcal{V}$-category: $\mathcal{V}$-cocomplete with a small strong generator consisting of finitely presentable objects;
- Finitely complete $\mathcal{V}$-category: one with finite conical limits and finite powers.

Theorem (Kelly)

The following is a biequivalence of 2-categories:

$$(-)^{op}_f : \mathcal{V}\text{-}\mathbf{Lfp} \rightleftarrows \mathcal{V}\text{-}\mathbf{Lex}^{op} : \mathrm{Lex}(-, \mathcal{V})$$

Base for Enrichment

Let $\mathcal{V} = (\mathcal{V}_0, I, \otimes)$ be a symmetric monoidal closed category.

Recall: An object A of $\mathcal{V}_0$ is called (regular) projective if the hom-functor $\mathcal{V}_0(A, -) : \mathcal{V}_0 \to \mathbf{Set}$ preserves regular epimorphisms; denote by $(\mathcal{V}_0)_{pf}$ the full subcategory of finite projective objects.

Definition

Let $\mathcal{V} = (\mathcal{V}_0, \otimes, I)$ be a symmetric monoidal closed category. We say that $\mathcal{V}$ is a symmetric monoidal finitary quasivariety if:

1. $\mathcal{V}_0$ is cocomplete with strong generator $\mathcal{P} \subseteq (\mathcal{V}_0)_{pf}$ (i.e. is a finitary quasivariety);
2. $I \in (\mathcal{V}_0)_f$;
3. if $P, Q \in \mathcal{P}$ then $P \otimes Q \in (\mathcal{V}_0)_{pf}$.

We call it a symmetric monoidal finitary variety if $\mathcal{V}_0$ is also a finitary variety (i.e. an exact finitary quasivariety).

Base for Enrichment

Examples

1. **Set**, **Ab**, R-Mod and GR-R-Mod, for each commutative ring R, with the usual tensor product;
2. $[\mathcal{C}^{op}, \mathbf{Set}]$, for any category $\mathcal{C}$ with finite products, equipped with the cartesian product;
3. pointed sets $\mathbf{Set}_*$ with the smash product;
4. G-sets $\mathbf{Set}^G$ for a finite group G with the cartesian product;
5. directed graphs **Gra** with the cartesian product;
6. Ch($\mathcal{A}$) for each abelian and symmetric monoidal finitary quasivariety $\mathcal{A}$, with the tensor product inherited from $\mathcal{A}$;
7. torsion free abelian groups $\mathbf{Ab}_{tf}$ with the usual tensor product;
8. binary relations **BRel** with the cartesian product;

Regular $\mathcal{V}$-categories

Definition

A $\mathcal{V}$-category $\mathcal{C}$ is called regular if:

- it has all finite weighted limits and coequalizers of kernel pairs;
- regular epimorphisms are stable under pullback and closed under powers by elements of $\mathcal{P} \subseteq (\mathcal{V}_0)_{pf}$.

$F : \mathcal{C} \to \mathcal{D}$ between regular $\mathcal{V}$-categories is called regular if it preserves finite weighted limits and regular epimorphisms.

- $\mathcal{V}$ itself is regular as a $\mathcal{V}$-category;
- if $\mathcal{C}$ is regular as a $\mathcal{V}$-category then $\mathcal{C}_0$ is a regular category;

Theorem (Barr's Embedding)

Let $\mathcal{C}$ be a small regular $\mathcal{V}$-category; then the evaluation functor $\mathrm{ev}_{\mathcal{C}} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathcal{V}), \mathcal{V}]$ is fully faithful and regular.

Regular $\mathcal{V}$-categories

Definition

A $\mathcal{V}$-category $\mathcal{C}$ is called regular if:

- it has all finite weighted limits and coequalizers of kernel pairs;
- regular epimorphisms are stable under pullback and closed under powers by elements of $\mathcal{P} \subseteq (\mathcal{V}_0)_{pf}$.

$F : \mathcal{C} \to \mathcal{D}$ between regular $\mathcal{V}$-categories is called regular if it preserves finite weighted limits and regular epimorphisms.

- $\mathcal{V}$ itself is regular as a $\mathcal{V}$-category;
- if $\mathcal{C}$ is regular as a $\mathcal{V}$-category then $\mathcal{C}_0$ is a regular category;

Theorem (Barr's Embedding)

Let $\mathcal{C}$ be a small regular $\mathcal{V}$-category; then the evaluation functor $\mathrm{ev}_{\mathcal{C}} : \mathcal{C} \to [\mathrm{Reg}(\mathcal{C}, \mathcal{V}), \mathcal{V}]$ *is fully faithful and regular.*

Exact $\mathcal{V}$-categories

Definition

A $\mathcal{V}$-category $\mathcal{B}$ is called exact if it is regular and in addition the ordinary category $\mathcal{B}_0$ is exact in the usual sense.

- Taking $\mathcal{V} = \mathbf{Set}$ or $\mathcal{V} = \mathbf{Ab}$ this notion coincides with the ordinary one of exact or abelian category.
- If $\mathcal{V}$ is a symmetric monoidal finitary variety, $\mathcal{V}$ is exact as a $\mathcal{V}$-category.

Theorem (Makkai's Image Theorem)

For any small exact $\mathcal{V}$-category $\mathcal{B}$; the essential image of $\mathrm{ev}_{\mathcal{B}} : \mathcal{B} \longrightarrow [\mathrm{Reg}(\mathcal{B}, \mathcal{V}), \mathcal{V}]$ is given by those functors which preserve small products, filtered colimits and projective powers.

Exact $\mathcal{V}$-categories

Definition

A $\mathcal{V}$-category $\mathcal{B}$ is called exact if it is regular and in addition the ordinary category $\mathcal{B}_0$ is exact in the usual sense.

- Taking $\mathcal{V} = \textbf{Set}$ or $\mathcal{V} = \textbf{Ab}$ this notion coincides with the ordinary one of exact or abelian category.
- If $\mathcal{V}$ is a symmetric monoidal finitary variety, $\mathcal{V}$ is exact as a $\mathcal{V}$-category.

Theorem (Makkai's Image Theorem)

For any small exact $\mathcal{V}$-category $\mathcal{B}$; the essential image of $\mathrm{ev}_{\mathcal{B}} : \mathcal{B} \longrightarrow [\mathrm{Reg}(\mathcal{B}, \mathcal{V}), \mathcal{V}]$ is given by those functors which preserve small products, filtered colimits and projective powers.

Definable $\mathcal{V}$-categories

Definition

- Given an arrow $h : A \to B$ in a $\mathcal{V}$-category $\mathcal{L}$, an object $L \in \mathcal{L}$ is said to be *h-injective* if $\mathcal{L}(h, L) : \mathcal{L}(B, L) \to \mathcal{L}(A, L)$ is a regular epimorphism in $\mathcal{V}$.
- Given a small set $\mathcal{M}$ of arrows from $\mathcal{L}$, write $\mathcal{M}$-inj for the full subcategory of $\mathcal{L}$ consisting of h-injective for each $h \in \mathcal{M}$.
- If $\mathcal{L}$ is locally finitely presentable and the arrows in $\mathcal{M}$ have finitely presentable domain and codomain, we call $\mathcal{M}$-inj an *enriched finite injectivity class*.

Proposition

Each finite injectivity class $\mathcal{D}$ of a locally finitely presentable $\mathcal{V}$-category $\mathcal{L}$ is closed under (small) products, projective powers, filtered colimits, and pure subobjects.

Definable $\mathcal{V}$-categories

Definition

- Given an arrow $h : A \to B$ in a $\mathcal{V}$-category $\mathcal{L}$, an object $L \in \mathcal{L}$ is said to be h-injective if $\mathcal{L}(h, L) : \mathcal{L}(B, L) \to \mathcal{L}(A, L)$ is a regular epimorphism in $\mathcal{V}$.
- Given a small set $\mathcal{M}$ of arrows from $\mathcal{L}$, write $\mathcal{M}$-inj for the full subcategory of $\mathcal{L}$ consisting of h-injective for each $h \in \mathcal{M}$.
- If $\mathcal{L}$ is locally finitely presentable and the arrows in $\mathcal{M}$ have finitely presentable domain and codomain, we call $\mathcal{M}$-inj an enriched finite injectivity class.

Proposition

Each finite injectivity class $\mathcal{D}$ of a locally finitely presentable $\mathcal{V}$-category $\mathcal{L}$ is closed under (small) products, projective powers, filtered colimits, and pure subobjects.

Definable $\mathcal{V}$-categories

Definition

- A $\mathcal{V}$-category $\mathcal{D}$ is called definable if it is an enriched finite injectivity class of some locally finitely presentable $\mathcal{V}$-category.
- A definable functor between definable $\mathcal{V}$-categories is a $\mathcal{V}$-functor that preserves products, projective powers, and filtered colimits.

- Each locally finitely presentable $\mathcal{V}$-category is definable;
- For any small regular $\mathcal{V}$-category $\mathcal{C}$, the $\mathcal{V}$-category $\mathrm{Reg}(\mathcal{C}, \mathcal{V})$ is definable. Indeed, $\mathrm{Reg}(\mathcal{C}, \mathcal{V}) = \mathcal{M}$-inj in $\mathrm{Lex}(\mathcal{C}, \mathcal{V})$, where

$$\mathcal{M} := \{\mathcal{C}(h, -) \mid h \text{ regular epimorphism in } \mathcal{C}\}.$$

Duality for Enriched Exact Categories

Assume $\mathcal{V}$ to be a symmetric monoidal finitary variety, then

- every definable $\mathcal{V}$-category $\mathcal{D}$ is equivalent $\mathrm{Reg}(\mathcal{B}, \mathcal{V})$ for a small exact $\mathcal{V}$-category $\mathcal{B}$;
- for each definable $\mathcal{D}$, the $\mathcal{V}$-category $\mathrm{Def}(\mathcal{D}, \mathcal{V})$ is small and exact.

This and Makkai's Image Theorem imply:

Theorem

Let $\mathcal{V}$ be a symmetric monoidal finitary variety. Then the 2-adjunction

$$\mathrm{Def}(-, \mathcal{V}) : \mathcal{V}\text{-}\mathbf{Def} \rightleftarrows \mathcal{V}\text{-}\mathbf{Ex}^{op} : \mathrm{Reg}(-, \mathcal{V})$$

is a biequivalence.

Duality for Enriched Exact Categories

Assume $\mathcal{V}$ to be a symmetric monoidal finitary variety, then

- every definable $\mathcal{V}$-category $\mathcal{D}$ is equivalent $\mathrm{Reg}(\mathcal{B}, \mathcal{V})$ for a small exact $\mathcal{V}$-category $\mathcal{B}$;
- for each definable $\mathcal{D}$, the $\mathcal{V}$-category $\mathrm{Def}(\mathcal{D}, \mathcal{V})$ is small and exact.

This and Makkai's Image Theorem imply:

Theorem

Let $\mathcal{V}$ be a symmetric monoidal finitary variety. Then the 2-adjunction

$$\mathrm{Def}(-, \mathcal{V}) : \mathcal{V}\text{-}\mathbf{Def} \rightleftarrows \mathcal{V}\text{-}\mathbf{Ex}^{op} : \mathrm{Reg}(-, \mathcal{V})$$

is a biequivalence.

Free Exact $\mathcal{V}$-categories

Proposition

Let $\mathcal{C}$ be a small finitely complete $\mathcal{V}$-category; then for each small exact $\mathcal{V}$-category $\mathcal{B}$, $\mathrm{ev} : \mathcal{C} \to \mathcal{C}_{ex/lex} := \mathrm{Def}(\mathrm{Lex}(\mathcal{C}, \mathcal{V}), \mathcal{V})$ *induces an equivalence:*

$$\mathrm{Reg}(\mathcal{C}_{ex/lex}, \mathcal{B}) \simeq \mathrm{Lex}(\mathcal{C}, \mathcal{B}).$$

and

Proposition

Let $\mathcal{C}$ be a small regular $\mathcal{V}$-category. Then for each small exact $\mathcal{V}$-category $\mathcal{B}$, $\mathrm{ev} : \mathcal{C} \to \mathcal{C}_{ex/reg} := \mathrm{Def}(\mathrm{Reg}(\mathcal{C}, \mathcal{V}), \mathcal{V})$ *induces an equivalence:*

$$\mathrm{Reg}(\mathcal{C}_{ex/reg}, \mathcal{B}) \simeq \mathrm{Reg}(\mathcal{C}, \mathcal{B}).$$

Thank You