On the category of cocommutative Hopf algebras

Florence Sterck

Joint work with Marino Gran and Joost Vercruysse

Université catholique de Louvain and Université Libre de Bruxelles

8 July 2019

Overview

Hopf algebra

A **Hopf algebra** H over a field K is given by

1. An algebra $(H,\, m : H \otimes H \to H,\, u : K \to H)$

$$\begin{tikzcd} H \otimes H \otimes H \arrow[r, "id_H \otimes m"] \arrow[d, "m \otimes id_H"'] & H \otimes H \arrow[d, "m"] \\ H \otimes H \arrow[r, "m"'] & H \end{tikzcd} \qquad \begin{tikzcd} & H \otimes H \arrow[d, "m"] & \\ K \otimes H \arrow[ur, "u \otimes id_H"] \arrow[r, "\cong"'] & H & H \otimes K \arrow[ul, "id_H \otimes u"'] \arrow[l, "\cong"] \end{tikzcd}$$

2. A coalgebra $(H,\, \Delta : H \to H \otimes H,\, \epsilon : H \to K)$

$$\begin{tikzcd} H \otimes H \otimes H & H \otimes H \arrow[l, "id_H \otimes \Delta"'] \\ H \otimes H \arrow[u, "\Delta \otimes id_H"] & H \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} & H \otimes H \arrow[dl, "\epsilon \otimes id_H"'] \arrow[dr, "id_H \otimes \epsilon"] & \\ K \otimes H & H \arrow[l, "\cong"] \arrow[r, "\cong"'] \arrow[u, "\Delta"'] & H \otimes K \end{tikzcd}$$

   We use Sweedler's notation, $\Delta(x) = x_1 \otimes x_2$.

3. Some conditions of compatibility
4. An antipode $S : H \to H$

Hopf algebra

A **Hopf algebra** H over a field K is given by

1. An algebra $m : H \otimes H \to H$, $u : K \to H$
2. A coalgebra $\Delta : H \to H \otimes H$, $\epsilon : H \to K$
3. Some conditions of compatibility,

$$\begin{tikzcd}
H \otimes H \arrow[r, "\Delta \otimes \Delta"] \arrow[d, "m"'] & H \otimes H \otimes H \otimes H \arrow[r, "1 \otimes \sigma \otimes 1"] & H \otimes H \otimes H \otimes H \arrow[d, "m \otimes m"] \\
H \arrow[rr, "\Delta"] & & H \otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
K \arrow[rr, equal] \arrow[dr, "u"'] & & K \\
& H \arrow[ur, "\epsilon"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
H \otimes H \arrow[rr, "m"] \arrow[dr, "\epsilon \otimes \epsilon"'] & & H \arrow[dl, "\epsilon"] \\
& K &
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[rr, "u \otimes u"] \arrow[dr, "u"'] & & H \otimes H \\
& H \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

where $\sigma(x \otimes y) = y \otimes x$.
$(H, m, u, \Delta, \epsilon)$ is a **bialgebra**.

4. An antipode $S : H \to H$

Hopf algebra

A **Hopf algebra** H over a field K is given by

1. A algebra $m : H \otimes H \to H$, $u : K \to H$
2. A coalgebra $\Delta : H \to H \otimes H$, $\epsilon : H \to K$
3. Some conditions of compatibility (bialgebra),
4. An antipode $S : H \to H$

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[drr, "\epsilon"'] & H \otimes H \arrow[r, shift left, "S \otimes id_H"] \arrow[r, shift right, "id_H \otimes S"'] & H \otimes H \arrow[r, "m"] & H \\ & & K \arrow[ur, "u"'] & \end{tikzcd}$$

A Hopf algebra H is called **cocommutative** if
$$\begin{tikzcd} H \arrow[rr, "\Delta"] \arrow[dr, "\Delta"'] & & H \otimes H \\ & H \otimes H \arrow[ur, "\sigma"'] & \end{tikzcd}$$
where $\sigma(x \otimes y) = y \otimes x$.

In Sweedler's notation : $x_1 \otimes x_2 = x_2 \otimes x_1$

Examples :

1. Let G be a group, $kG = \{\sum_g \alpha_g g | g \in G\}$ the group algebra is a Hopf algebra,

$$\Delta(g) = g \otimes g, \;\; \epsilon(g) = 1, \;\; S(g) = g^{-1}$$

2. Let $\mathfrak{g}$ be a Lie algebra, $U(\mathfrak{g})$ is a Hopf algebra with

$$\Delta(x) = 1 \otimes x + x \otimes 1, \;\; \epsilon(x) = 0, \;\; S(x) = -x.$$

$\mathsf{Hopf}_{K,coc}$ objects : cocommutative Hopf algbras
arrows : morphisms of Hopf algebras i.e. morphisms of algebras and coalgebras

$$\begin{tikzcd} H \otimes H \arrow[r, "f \otimes f"] \arrow[d, "m"'] & H' \otimes H' \arrow[d, "m"] \\ H \arrow[r, "f"'] & H \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "u"] \arrow[dr, "u"'] & H \arrow[d, "f"] \\ & H' \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "f"'] & H \otimes H \arrow[d, "f \otimes f"] \\ H' \arrow[r, "\Delta"'] & H' \otimes H' \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "f"] \arrow[dr, "\epsilon"'] & H' \arrow[d, "\epsilon"] \\ & K \end{tikzcd}$$

Semi-abelian

Definition (Janelidze, Marki, Tholen (2002, JPAA))

A category $\mathcal{C}$ is **semi-abelian** if and only if

1. pointed
2. regular
   1. finitely complete
   2. regular epi/mono factorization
   3. pullback stability of regular epimorphisms
3. protomodular
4. exact
5. binary coproducts

Examples : Grp, Lie_K, CompGrp, ...

$\mathrm{Hopf}_{K,coc}$ is semi-abelian

1. pointed **i.e.** $\exists$ **a zero object,** 0**, such that** $\forall X \in \mathcal{C}$, $\exists!\ X \to 0$ **and** $0 \to X$

   In $\mathrm{Hopf}_{\mathbf{K},\mathbf{coc}}$**, the base field K is the zero object, with** ϵ **and** u**.**
2. regular
   1. finitely complete
   2. regular epi/mono factorization
   3. pullback stability of regular epimorphisms
3. protomodular
4. exact
5. binary coproducts

$\mathrm{Hopf}_{K,coc}$ is semi-abelian

1. pointed
2. regular
   1. finitely complete **finite products and equalizers**

$$\begin{tikzcd} & A \otimes B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & \psi & B \\ & X \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, dashed] & \end{tikzcd}$$

$$\pi_A = Id \otimes \epsilon$$
$$\pi_B = \epsilon \otimes Id$$

$$\psi(x) = f(x_1) \otimes g(x_2).$$

$$\Delta(\psi(x)) = f(x_1)_1 \otimes g(x_2)_1 \otimes f(x_1)_1 \otimes g(x_2)_2$$
$$= f(x_1) \otimes g(x_3) \otimes f(x_2) \otimes g(x_4)$$
$$(\psi \otimes \psi) \cdot \Delta = f(x_1) \otimes g(x_2) \otimes f(x_3) \otimes g(x_4)$$

The equalizer of $\mathbf{f}, \mathbf{g} : \mathbf{A} \to \mathbf{B}$ is given by
$Eq(f, g) = \{a \in A \mid a_1 \otimes f(a_2) \otimes a_3 = a_1 \otimes g(a_2) \otimes a_3\}.$

   2. regular epi/mono factorization
   3. pullback stability of regular epimorphisms
3. protomodular
4. exact
5. binary coproducts

$\mathsf{Hopf}_{K,coc}$ is semi-abelian

1. pointed
2. regular
    1. finitely complete
    2. regular epi/mono factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "f"'] & & B \\ & f(A) \arrow[ur, "inc"'] & \end{tikzcd}$$

In $\mathsf{Hopf}_{K,coc}$**,**
regular epimorphisms = surjective morphisms

    3. pullback stability of regular epimorphisms
3. protomodular
4. exact
5. binary coproducts

$\mathrm{Hopf}_{K,coc}$ is semi-abelian

1. pointed
2. regular
   1. finitely complete
   2. regular epi/mono factorization
   3. pullback stability of regular epimorphisms

$$\begin{tikzcd} A \times_B C \arrow[r, dashed, two heads, "\pi_C"] \arrow[d, "\pi_A"'] & C \arrow[d] \\ A \arrow[r, two heads, "f"'] & B \end{tikzcd}$$

To prove it we use a result of Newman, there is a bijection between Hopf subalgebras and left ideals and two-sided coideals

3. protomodular
4. exact
5. binary coproducts

$\mathrm{Hopf}_{K,coc}$ is semi-abelian

1. pointed
2. regular
    1. finitely complete
    2. regular epi/mono factorization
    3. pullback stability of regular epimorphisms
3. protomodular

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "k"] \arrow[d, "v"] & A \arrow[r, shift right, "f"'] \arrow[d, dashed, "u"] & A \arrow[l, shift right, "s"'] \arrow[d, "w"] \\
0 \arrow[r] & K' \arrow[r, "k'"'] & A' \arrow[r, shift right, "f'"'] & A' \arrow[l, shift right, "s'"']
\end{tikzcd}$$

$$\mathrm{Hopf}_{K,coc} = \mathrm{Grp}(\mathrm{CoAlg}_{K,coc})$$

4. exact
5. binary coproducts

$\mathrm{Hopf}_{K,coc}$ is semi-abelian

1. pointed
2. regular
   1. finitely complete
   2. regular epi/mono factorization
   3. pullback stability of regular epimorphisms
3. protomodular
4. exact :

   Since we have pointed, regular and protomodular

$$\begin{tikzcd} N \arrow[r, two heads, "f"] \arrow[d, tail] & f(N) \arrow[d, dashed, tail] \\ H \arrow[r, two heads, "f"'] & G \end{tikzcd}$$

   $\mathbf{N \to H}$ is normal iff $\mathbf{h_1 n S(h_2) \in N}$ $\forall \mathbf{h \in H, n \in N}$. f surjective

$$\begin{aligned} g_1 f(n) S(g_2) &= f(h_1) f(n) f(S(h_2)) \\ &= f(h_1 n S(h_2)) \in f(N) \end{aligned}$$

   where $\mathbf{f(h) = g}$.

5. binary coproducts

$\mathrm{Hopf}_{K,coc}$ is semi-abelian

1. pointed
2. regular
   1. finitely complete
   2. regular epi/mono factorization
   3. pullback stability of regular epimorphisms
3. protomodular
4. exact
5. binary coproducts **as in the category of algebras.**

Theorem (Gran, Sterck, Vercruysse (2019, JPAA))

$\mathsf{Hopf}_{K,coc}$ *is semi-abelian.*

Theorem (Gran, Sterck, Vercruysse (2019, JPAA))

$\mathsf{Hopf}_{K,coc}$ *is semi-abelian.*

Consequences :

1. Noether's isomorphism theorems
2. classical homological lemmas
3. commutator theory
4. categorical notion of action, semi-direct product and **crossed modules**

Theorem (Gran, Sterck, Vercruysse (2019, JPAA))

$\mathsf{Hopf}_{\mathsf{K,coc}}$ *is semi-abelian.*

Consequences :

1. Noether's isomorphism theorems
2. classical homological lemmas
3. commutator theory
4. categorical notion of action, semi-direct product and **crossed modules**

Theorem (Janelidze (2003, GMJ))

If $\mathcal{C}$ is a semi-abelian category, then

$$\mathsf{XMod}(\mathcal{C}) \cong \mathsf{Grpd}(\mathcal{C})$$

Crossed modules of groups

Crossed modules

$\mu : A \to B$ a group morphism,
A a B-group, $B \times A \to A$, such that

$$\mu({}^{b}a) = b\mu(a)b^{-1},$$
$$^{\mu(a)}a' = aa'a^{-1}.$$

Internal groupoids in Grp

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & G_0 \arrow[l, "e" description] \end{tikzcd}$$

where s, t, e, i are the "source", "target", "identity", "inverse" morphisms, and m is the multiplication/composition of "composable" morphisms.

Crossed modules of groups

Crossed modules

$\mu : A \to B$ a group morphism,
A a B-group, $B \times A \to A$, such that

$$\mu({}^{b}a) = b\mu(a)b^{-1},$$
$$^{\mu(a)}a' = aa'a^{-1}.$$

Internal groupoids in Grp

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & G_0 \arrow[l, "e" description] \end{tikzcd}$$

$$\begin{tikzcd} (A \rtimes B) \times_B (A \rtimes B) \arrow[r, "m"] & A \rtimes B \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & B \arrow[l, "e" description] \end{tikzcd}$$

where

$$m((a,b),(a',b')) = (aa', b');$$
$$s(a,b) = b;$$
$$t(a,b) = \mu(a)b;$$
$$e(b) = (1_A, b).$$

Crossed modules of groups

Crossed modules

$\mu : A \to B$ a group morphism,
A a B-group, $B \times A \to A$, such that

$$\mu({}^{b}a) = b\mu(a)b^{-1},$$
$$^{\mu(a)}a' = aa'a^{-1}.$$

Internal groupoids in Grp

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "m"] & G_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & G_0 \arrow[l, "e" description] \end{tikzcd}$$

$$\mu := t|_{Ker(s)} : Ker(s) \to G_0;$$
$$G_0 \times Ker(s) \to Ker(s) : (g, k) \to e(g)ke(g)^{-1}.$$

Hopf crossed modules

Hopf crossed modules, Fernandez Vilaboa, Lopèz Lopèz and Villanueva Novoa (2006, CA), Majid (2012, ArXiv)

$d : X \to H$ a morphism of Hopf algebras, X a H-module Hopf algebra, $H \otimes X \to X$, such that

$$d({}^{h}x) = h_1 d(x) S(h_2),$$
$${}^{d(a)}x = a_1 x S(a_2).$$

Internal groupoids in $\mathrm{Hopf}_{K,coc}$

$$\begin{tikzcd} H_1 \times_{H_0} H_1 \arrow[r, "m"] & H_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & H_0 \arrow[l, "e" description] \end{tikzcd}$$

where s, t, e, i are the "source", "target", "identity", "inverse" morphisms, and m is the multiplication/composition of "composable" morphisms.

Hopf crossed modules

Hopf crossed modules

$d : X \to H$ a morphism of Hopf algebras, X a H-module Hopf algebra, $H \otimes X \to X$, such that

$$d({}^{h}x) = h_1 d(x) S(h_2),$$
$${}^{d(a)}x = a_1 x S(a_2).$$

Internal groupoids in $\mathsf{Hopf}_{K,coc}$

$$\begin{tikzcd} H_1 \times_{H_0} H_1 \arrow[r, "m"] & H_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & H_0 \arrow[l, "e"' description] \end{tikzcd}$$

$$\begin{tikzcd} (X \rtimes H) \times_H (X \rtimes H) \arrow[r, "m"] & X \rtimes H \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & H \arrow[l, "e" description] \end{tikzcd}$$

where
$$\begin{aligned} m((x \otimes h), (x' \otimes h')) &= (xx', \epsilon(h)h'); \\ s(x \otimes h) &= \epsilon(x)h; \\ t(x \otimes h) &= d(x)h; \\ e(h) &= 1_X \otimes h. \end{aligned}$$

Hopf crossed modules

Hopf crossed modules

$d : X \to H$ a morphism of Hopf algebras, X a H-module Hopf algebra, $H \otimes X \to X$, such that

$$d({}^{h}x) = h_1 d(x) S(h_2),$$
$${}^{d(a)}x = a_1 x S(a_2).$$

Internal groupoids in $\mathsf{Hopf}_{K,coc}$

$$\begin{tikzcd} H_1 \times_{H_0} H_1 \arrow[r, "m"] & H_1 \arrow[loop above, "i"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & H_0 \arrow[l, "e"'] \end{tikzcd}$$

$$d := t|_{HKer(s)} : HKer(s) \to H_0,$$
$$H_0 \otimes HKer(s) \to HKer(s) : h \otimes k \to e(h_1) k e(S(h_2)).$$

Hopf crossed modules

Theorem

$$\mathrm{HXMod}_{K,coc} \cong \mathrm{Grpd}(\mathrm{Hopf}_{\mathrm{K}}, \mathrm{coc})$$

$$\mathrm{HXMod}_{K,coc} \cong \mathrm{XMod}(\mathrm{Hopf}_{\mathrm{K}}, \mathrm{coc})$$

The notion of Hopf crossed modules and the one given by the construction of Janelidze coincide.

Commutator

In any pointed category $\mathcal{C}$ with binary products, two subobjects $x \colon X \to A$ and $y \colon Y \to A$ *commute* (in the sense of Huq) if and only if there exists an arrow p making the following diagram commute :

$$\begin{tikzcd} X \arrow[r, "{(1,0)}"] \arrow[dr, "x"'] & X \times Y \arrow[d, dashed, "p"] & Y \arrow[l, "{(0,1)}"'] \arrow[dl, "y"] \\ & A & \end{tikzcd} \tag{1}$$

In $\mathrm{Hopf}_{K,coc}$, the following conditions are equivalent :

(a) $\exists!$ morphism of Hopf algebras $p : X \otimes Y \to A$ such that diagram (1) commutes ;

(b) $ab = ba$, $\forall a \in X$ and $\forall b \in Y$;

(c) $a_1 b_1 S(a_2) S(b_2) = \epsilon(a)\epsilon(b)$, $\forall a \in X$ and $\forall b \in Y$.

Proposition

In $\mathrm{Hopf}_{K,coc}$, *let* X, Y *be normal Hopf subalgebras of* A, $[X,Y]_{Huq}$ *is the algebra generated by*

$$x_1 y_1 S(x_2) S(y_2)$$

This commutator coincides with the one given by Yanagihara (1978, JA).

Proposition

In $\mathsf{Hopf}_{K,coc}$*, let* X*,* Y *be normal Hopf subalgebras of* A*,* $[X,Y]_{Huq}$ *is the algebra generated by*

$$x_1 y_1 S(x_2) S(y_2)$$

This commutator coincides with the one given by Yanagihara (1978, JA).

In $\mathsf{Hopf}_{K,coc}$, the following categories are equivalent

1. $\mathsf{Grpd}(\mathsf{Hopf}_{\mathsf{K,coc}})$
2. $\mathsf{HXmod}(\mathsf{Hopf}_{\mathsf{K,coc}})$
3. $\mathsf{Cat}^1(\mathsf{Hopf}_{\mathsf{K,coc}})$

where a cat^1-Hopf algebra is a reflexive graph
$$\begin{tikzcd} H_1 \arrow[r, shift left=3, "s"] \arrow[r, shift right=3, "t"'] & H_0 \arrow[l, "e"' description] \end{tikzcd}$$
such that

$[HKer(s), HKer(t)] = 0$ i.e. $kh = hk \ \forall h \in HKer(s), k \in HKer(t)$

Work in progress, a definition of *Hopf crossed square* such that

$$Grpd^2(\mathsf{Hopf}_{\mathsf{K,coc}}) \cong \mathsf{Cat}^2(\mathsf{Hopf}_{\mathsf{K,coc}}) \cong \mathsf{X}^2(\mathsf{Hopf}_{\mathsf{K,coc}})$$

Bibliography

G. Böhm,
Crossed modules of monoids II, preprint, arXiv :1803.04124v1, 2018.

M. Gran, G. Kadjo, and J. Vercruysse,
A torsion theory in the category of cocommutative Hopf algebras.
Appl. Categ. Struct., 24, 269-282, 2016.

J.M. Fernández Vilaboa, M.P. López López and E. Villanueva Novoa,
Cat^1-Hopf Algebras and Crossed Modules,
Commun. Algebra 35 (1) 181-191, 2006.

G. Janelidze,
Internal crossed modules.
Georgian Math. Journal 10 (1), 99-114, 2003.

S. Majid,
Strict quantum 2-groups, Preprint, https ://arxiv.org/abs/1208.6265, 2012.

H. Yanagihara,
On isomorphism theorems of formal groups,
J. Algebra 55, 341–347, 1978.