On the category of cocommutative Hopf algebras

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Overview

- Hopf algebras
- 2 Semi-abelian
- Crossed modules
- Commutator

Hopf algebra

A **Hopf** algebra H over a field K is given by

1 An algebra $(H, m: H \otimes H \rightarrow H, u: K \rightarrow H)$

$$\begin{array}{c} H \otimes H \otimes H \stackrel{id_H \otimes m}{\longrightarrow} H \otimes H \\ m \otimes id_H \downarrow & \downarrow m \\ H \otimes H \stackrel{\longrightarrow}{\longrightarrow} H \end{array}$$

② A coalgebra $(H, \Delta : H \rightarrow H \otimes H, \epsilon : H \rightarrow K)$

We use Sweedler's notation, $\Delta(x) = x_1 \otimes x_2$.

- Some conditions of compatibility
- An antipode $S: H \to H$

Hopf algebra

A **Hopf algebra** H over a field K is given by

- **3** A coalgebra $\Delta: H \to H \otimes H$, $\epsilon: H \to K$
- Some conditions of compatibility,

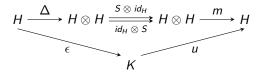
where
$$\sigma(x \otimes y) = y \otimes x$$
. $(H, m, u, \Delta, \epsilon)$ is a **bialgebra**.

a An antipode $S: H \rightarrow H$

Hopf algebra

A **Hopf algebra** H over a field K is given by

- ② A coalgebra $\Delta: H \to H \otimes H$, $\epsilon: H \to K$
- Some conditions of compatibility (bialgebra),
- **1** An antipode $S: H \rightarrow H$



A Hopf algebra H is called **cocommutative** if $H \xrightarrow{\Delta} H \otimes H$ where $\sigma(x \otimes y) = y \otimes x$.

In Sweedler's notation : $x_1 \otimes x_2 = x_2 \otimes x_1$

Examples:

① Let G be a group, $kG = \{\sum_g \alpha_g g | g \in G\}$ the group algebra is a Hopf algebra,

$$\Delta(g) = g \otimes g, \ \epsilon(g) = 1, \ S(g) = g^{-1}$$

② Let \mathfrak{g} be a Lie algebra, $U(\mathfrak{g})$ is a Hopf algebra with

$$\Delta(x) = 1 \otimes x + x \otimes 1, \ \epsilon(x) = 0, \ S(x) = -x.$$

Hopf_{K,coc} objects : cocommutative Hopf algbras

arrows: morphisms of Hopf algebras i.e. morphisms of algebras and coalgebras

$$\begin{array}{ccc}
H \otimes H \xrightarrow{f \otimes f} H' \otimes H' \\
m \downarrow & \downarrow m \\
H \xrightarrow{f} & H
\end{array}$$





Semi-abelian

Definition (Janelidze, Marki, Tholen (2002, JPAA))

A category C is **semi-abelian** if and only if

- pointed
- regular
 - finitely complete
 - regular epi/mono factorization
 - pullback stability of regular epimorphisms
- protomodular
- exact
- binary coproducts

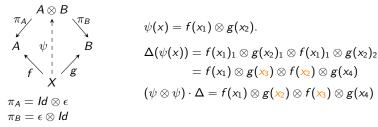
Examples: Grp, Lie_K, CompGrp, ...



- **9** pointed i.e. \exists a zero object, 0, such that $\forall X \in \mathcal{C}$, $\exists ! X \to 0$ and $0 \to X$
 - In Hopf_{K coc}, the base field K is the zero object, with ϵ and u.
- regular
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- pointed
- egular
 - finitely complete finite products and equalizers



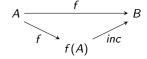
The equalizer of $f,g:\boldsymbol{A}\to\boldsymbol{B}$ is given by

$$Eq(f,g) = \{a \in A \mid a_1 \otimes f(a_2) \otimes a_3 = a_1 \otimes g(a_2) \otimes a_3\}.$$

- regular epi/mono factorization
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- pointed
- egular
 - finitely complete
 - regular epi/mono factorization



 $\begin{array}{l} \text{In } \mathsf{Hopf}_{K,coc}, \\ \mathbf{regular \ epimorphisms} = \mathbf{surjective} \\ \mathbf{morphisms} \end{array}$

- 3 pullback stability of regular epimorphisms
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- pointed
- 2 regular
 - finitely complete
 - regular epi/mono factorization
 - pullback stability of regular epimorphisms

$$\begin{array}{ccc}
A \times_B C & \xrightarrow{\pi_C} & \times C \\
\pi_A \downarrow & & \downarrow \\
A & \xrightarrow{f} & B
\end{array}$$

To prove it we use a result of Newman, there is a bijection between Hopf subalgebras and left ideals and two-sided coideals

- protomodular
- exact
- binary coproducts



- pointed
- regular
 - finitely complete
 - g regular epi/mono factorization
 - 9 pullback stability of regular epimorphisms
- protomodular

$$0 \longrightarrow K \xrightarrow{k} A \xrightarrow{s} A$$

$$\downarrow v \qquad \qquad \downarrow w \qquad \text{Hopf}_{K,coc} = \text{Grp}(\text{CoAlg}_{K,coc})$$

$$0 \longrightarrow K' \xrightarrow{k'} A' \xrightarrow{f'} A'$$

- exact
- binary coproducts



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 - 9 pullback stability of regular epimorphisms
- protomodular
- exact :

Since we have pointed, regular and protomodular

$$\begin{array}{ccc}
N & \xrightarrow{f} & f(N) \\
 & & & \\
 & & \downarrow \\
 & & H & \xrightarrow{f} & G
\end{array}$$

binary coproducts

 $N \to H$ is normal iff $h_1 n S(h_2) \in N$ $\forall h \in H, n \in N$. f surjective

$$g_1 f(n) S(g_2) = f(h_1) f(n) f(S(h_2))$$

= $f(h_1 n S(h_2)) \in f(N)$

where f(h) = g.

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- pointed
- regular
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- exact
- **5** binary coproducts as in the category of algebras.



Theorem (Gran, Sterck, Vercruysse (2019, JPAA))

 $\mathsf{Hopf}_{\mathsf{K},\mathsf{coc}}$ is semi-abelian.



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Consequences:

- Noether's isomorphism theorems
- classical homological lemmas
- commutator theory
- categorical notion of action, semi-direct product and crossed modules

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Theorem (Janelidze (2003, GMJ))

If C is a semi-abelian category, then

$$\mathsf{XMod}(\mathcal{C}) \cong \mathsf{Grpd}(\mathcal{C})$$

Crossed modules of groups

Crossed modules

 $\mu:A\to B$ a group morphism, A a B-group, $B\times A\to A$, such that

$$\mu(b^a) = b\mu(a)b^{-1},$$

 $\mu(a)a' = aa'a^{-1}.$

Internal groupoids in Grp

$$G_1 \times_{G_0} G_1 \xrightarrow{m} G_1 \xrightarrow{s} G_0$$

where s, t, e, i are the "source", "target", "identity", "inverse" morphisms, and m is the multiplication/composition of "composable" morphisms.

Crossed modules of groups

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$$\mu({}^b a) = b\mu(a)b^{-1},$$

 $\mu({}^a)a' = aa'a^{-1}.$

$(A \rtimes B) \times_B (A \rtimes B) \xrightarrow{m} A \rtimes B \xrightarrow{s} B$

where

$$m((a,b),(a',b')) = (aa',b');$$

 $s(a,b) = b;$
 $t(a,b) = \mu(a)b;$
 $e(b) = (1_A,b).$

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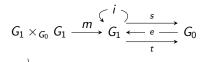
$$\mu({}^ba) = b\mu(a)b^{-1},$$

 $\mu({}^a)a' = aa'a^{-1}.$

$\mu:=t|_{\mathit{Ker}(s)}:\mathit{Ker}(s) o \mathit{G}_{0};$

$$G_0 \times \operatorname{{\it Ker}}(s) \to \operatorname{{\it Ker}}(s) : (g,k) \to \operatorname{{\it e}}(g)\operatorname{{\it ke}}(g)^{-1}.$$

Internal groupoids in Grp



Hopf crossed modules, Fernandez Vilaboa, Lopèz Lopèz and Villanueva Novoa (2006, CA), Majid (2012, ArXiv)

 $d: X \to H$ a morphism of Hopf algebras, X a H-module Hopf algebra, $H \otimes X \to X$, such that

$$d(^{h}x) = h_1 d(x)S(h_2),$$

 $d(a)x = a_1xS(a_2).$

Internal groupoids in $Hopf_{K,coc}$

$$H_1 \times_{H_0} H_1 \xrightarrow{m} H_1 \xrightarrow{s} H_0$$

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Internal groupoids in $Hopf_{K,coc}$

$$H_1 \times_{H_0} H_1 \xrightarrow{m} H_1 \xrightarrow{s} H_0$$

$$(X \times H) \times_H (X \times H) \xrightarrow{m} X \times H \xrightarrow{s} H$$

$$m((x \otimes h), (x' \otimes h')) = (xx', \epsilon(h)h');$$

$$s(x \otimes h) = \epsilon(x)h;$$

$$t(x \otimes h) = d(x)h;$$

$$e(h) = 1_X \otimes h.$$

where

Hopf crossed modules

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 $^{d(a)}x = a_1 x S(a_2).$

Internal groupoids in $Hopf_{K,coc}$

$$H_1 \times_{H_0} H_1 \xrightarrow{m} H_1 \xrightarrow{\epsilon} H_0$$

$$d:=t|_{HKer(s)}:HKer(s)\to H_0,$$

$$H_0 \otimes HKer(s) \rightarrow HKer(s) : h \otimes k \rightarrow e(h_1)ke(S(h_2)).$$

Theorem

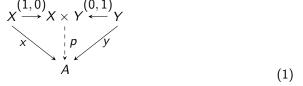
$$\mathsf{HXMod}_{K,coc} \cong \mathsf{Grpd}(\mathsf{Hopf}_{\mathsf{K}},\mathsf{coc})$$

$$\mathsf{HXMod}_{K,coc} \cong \mathsf{XMod}(\mathsf{Hopf}_\mathsf{K},\mathsf{coc})$$

The notion of Hopf crossed modules and the one given by the construction of Janelidze coincide.

Commutator

In any pointed category $\mathcal C$ with binary products, two subobjects $x\colon X\to A$ and $y\colon Y\to A$ commute (in the sense of Huq) if and only if there exists an arrow p making the following diagram commute :



In $\mathsf{Hopf}_{K,coc}$, the following conditions are equivalent :

- (a) $\exists !$ morphism of Hopf algebras $p: X \otimes Y \to A$ such that diagram (1) commutes;
- (b) ab = ba, $\forall a \in X$ and $\forall b \in Y$;
- (c) $a_1b_1S(a_2)S(b_2) = \epsilon(a)\epsilon(b)$, $\forall a \in X$ and $\forall b \in Y$.



Proposition

In $\mathsf{Hopf}_{K,\mathsf{coc}}$, let X, Y be normal Hopf subalgebras of A, $[X,Y]_{\mathsf{Huq}}$ is the algebra generated by

$$x_1y_1S(x_2)S(y_2)$$

This commutator coincides with the one given by Yanagihara (1978, JA).



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In $\mathsf{Hopf}_{K,coc}$, the following categories are equivalent

- Grpd(Hopf_{K,coc})
- 4 HXmod(Hopf_{K,coc})
- Cat¹(Hopf_{K,coc})

where a cat¹-Hopf algebra is a reflexive graph $H_1 \xrightarrow[t]{\overset{\circ}{\longleftarrow} e} H_0$ such that

$$[HKer(s), HKer(t)] = 0$$
 i.e. $kh = hk \ \forall h \in HKer(s), k \in HKer(t)$



Work in progress, a definition of Hopf crossed square such that

$$\mathit{Grpd}^2(\mathsf{Hopf}_{\mathsf{K},\mathsf{coc}}) \cong \mathsf{Cat}^2(\mathsf{Hopf}_{\mathsf{K},\mathsf{coc}}) \cong \mathsf{X}^2(\mathsf{Hopf}_{\mathsf{K},\mathsf{coc}})$$



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