Accessible aspects of 2-category theory

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Plan

1. Locally presentable categories and accessible categories.

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2. Two dimensional universal algebra.

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3. A general approach to accessibility of weak/cofibrant categorical structures.

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4. Quasicategories and related structures (w' Lack/Vokřínek).

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- Very nice: easy to construct adjoint functors between as solution set condition easy to verify. Stable under lots of limit constructions.
- Interested in the world in between accessible and locally presentable! E.g. weakly locally λ-presentable: λ-accessible and products/weak colimits. (AR1990s)

Two dimensional universal algebra – Sydney 1980s

- Two-dimensional universal algebra: e.g. 2-category $\mathbf{MonCat}_p$ of monoidal categories and strong monoidal functors: $f(a \otimes b) \cong fa \otimes fb$ and $f(i) \cong i$.

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- Today, we'll see such 2-cats are moreover accessible.

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- Visited Makkai in Budapest 2015 and chatted about all of this.
- Will describe general approach to accessibility of weak objects and weak maps. Some parts worked out by Makkai and some by me.

Makkai's generalised sketches 2 – terminal objects

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- Then X_T is set of all terminal objects, so $TOb_p \hookrightarrow Sk$ is the full subcat of injectives, so accessible.

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- For $\mathcal{C} \in \mathbf{K}$ we say that $\mathcal{C} \in \mathbf{K}^+$ if the full subcategory $RE(\mathcal{C}) \to Arr(\mathcal{C})$ of retract equivalences in $\mathcal{C}$ is accessible and accessibly embedded in the arrow category of $\mathcal{C}$.

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Proposition

$\mathbf{K}^+$ *is closed in* 2-Cat *under bilimits – in particular, pullbacks of isofibrations.*

Cellular 2-categories

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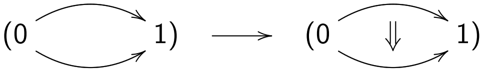
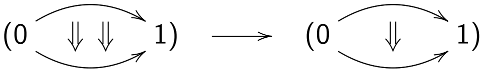
- $\delta D_2 \to D_2$: 

- $\delta D_3 \to D_3$: 

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Let $\mathcal{C} \in \mathbf{K}^+$. Then $Ps(D_i, \mathcal{C}) \to Ps(\delta D_i, \mathcal{C}) \in \mathbb{K}^+$ for $i = 0, 1, 2, 3$ and each such 2-category has flexible limits and filtered colimits pointwise.

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Using accessibility of these notions, we deduce result. □

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{\scriptstyle U_1}\downarrow & & \downarrow{\scriptstyle Ps(j_1, \mathbf{Cat})} \\
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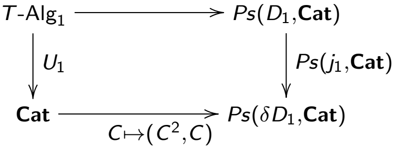

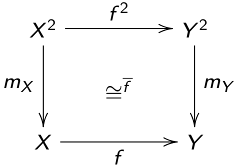

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- Right leg isofibration in $\mathbb{K}^+$. Bottom leg preserves limits and filtered colimits, and so belongs to $\mathbb{K}^+.\mathbb{K}^+$ closed in 2-Cat under pullbacks of isofibrations – hence $T\text{-Alg}_1 \to \mathbf{Cat} \in \mathbb{K}^+$.

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- Add associators by forming a pullback

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Here R sends (C, m) to the two paths from C^3 to C as on the right above.

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- Conclude that $\mathbf{MonCat}_p$ belongs to $\mathbb{K}^+$.

More examples and results using cocellularity

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- Also more general results for finite limit 2-theories. ...

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- Open problem: understand accessiblity of weak objects and weak maps in more contexts. E.g. when is the Kleisli category for a comonad accessible?

Final thoughts!

- Paper "Accessible aspects of 2-category theory" in the coming months, if you are interested.

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- Thanks for listening!

References

AR Weakly locally presentable categories, Adamek and Rosicky, 1994.

BKP : Two-dimensional monad theory, Blackwell, Kelly and Power, 1989.

BKPS : Flexible limits . . . , Bird, Kelly, Power and Street, 1989.

GU : Lokal Prsentierbare Kategorien, Gabriel and Ulmer, 1971.

MP : Accessible categories: the foundations of categorical model theory, Makkai and Pare, 1989.

M : Generalized sketches . . . , Makkai, 1997.

LR : Enriched weakness, Lack and Rosicky, 2012.

RV : Elements of infinity category theory, Riehl and Verity, in preparation.