# A Solution for the Compositionality Problem of Dinatural Transformations

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# Dinatural transformations

$F, G\colon \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{D}$. A *dinatural transformation* $\varphi\colon F \to G$ is a family of morphisms in $\mathbb{D}$

$$\varphi = (\varphi_A\colon F(A,A) \to G(A,A))_{A \in \mathbb{C}}$$

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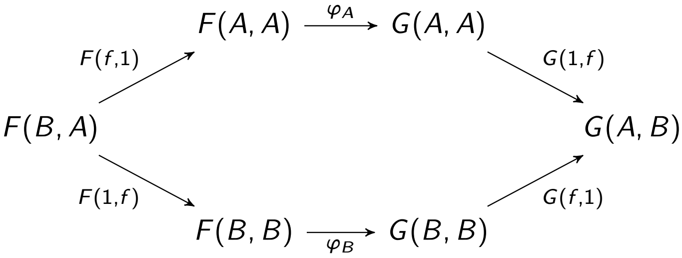
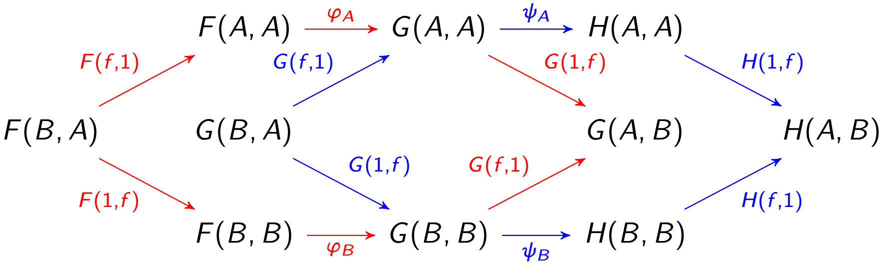
such that for all $f\colon A \to B$ in $\mathbb{C}$ the following commutes:



# . . . don't compose

$\varphi\colon F \to G$, $\psi\colon G \to H$ dinatural



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*eval* is natural in $B$ and for all $f \colon A \to A'$ the following commutes:

$$\begin{array}{ccc} A \times (A' \Rightarrow B) & \xrightarrow{1 \times (f \Rightarrow 1)} & A \times (A \Rightarrow B) \\ {\scriptstyle f \times (1 \Rightarrow 1)} \downarrow & & \downarrow {\scriptstyle eval_{A,B}} \\ A' \times (A' \Rightarrow B) & \xrightarrow{eval_{A',B}} & B \end{array}$$

since for all $a \in A$ and $g \colon A' \to B$ $(g \circ f)(a) = g(f(a))$.

# Extranatural transformations (Eilenberg, Kelly 1966)

$F\colon \mathbb{A} \times \mathbb{B}^{\mathsf{op}} \times \mathbb{B} \to \mathbb{E}$, $G\colon \mathbb{A} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{E}$. An *extranatural* transformation $\varphi\colon F \to G$ is a family of morphisms in $\mathbb{E}$

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such that for all $f\colon A \underset{\mathbb{A}}{\to} A'$, $g\colon B \underset{\mathbb{B}}{\to} B'$, $h\colon C \underset{\mathbb{C}}{\to} C'$

$$\begin{array}{ccc} F(A,B,B) & \xrightarrow{\varphi_{A,B,C}} & G(A,C,C) \\ {\scriptstyle F(f,1,1)}\downarrow & & \downarrow{\scriptstyle G(f,1,1)} \\ F(A',B,B) & \xrightarrow{\varphi_{A',B,C}} & G(A',C,C) \end{array} \qquad \begin{array}{ccc} F(A,B',B) & \xrightarrow{F(1,g,1)} & F(A,B,B) \\ {\scriptstyle F(1,1,g)}\downarrow & & \downarrow{\scriptstyle \varphi_{A,B,C}} \\ F(A,B',B') & \xrightarrow{\varphi_{A,B',C}} & G(A,C,C) \end{array}$$

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# Extranaturals don't compose already

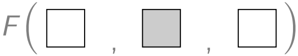
$F\colon \mathbb{A} \times \mathbb{B}^{\mathsf{op}} \times \mathbb{B} \to \mathbb{E}$, $G\colon \mathbb{A} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{E}$, $H\colon \mathbb{A} \times \mathbb{D}^{\mathsf{op}} \times \mathbb{D} \to \mathbb{E}$.
$\varphi\colon F \to G$, $\psi\colon G \to H$ extranatural transformations.

$$\psi \circ \varphi = \left( \; F(A,B,B) \xrightarrow{\varphi_{A,B,C}} G(A,C,C) \xrightarrow{\psi_{A,C,D}} H(A,D,D) \; \right)_{A,B,C,D}$$

is not a well-defined extranatural transformation from $F$ to $H$.

# A string diagrammatic calculus

$F\colon \mathbb{A} \times \mathbb{B}^{\mathsf{op}} \times \mathbb{B} \to \mathbb{E}$, $G\colon \mathbb{A} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{E}$



$\varphi = (\varphi_{A,B,C}\colon F(A,B,B) \to G(A,C,C))_{A,B,C} \rightsquigarrow$

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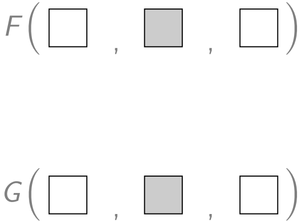

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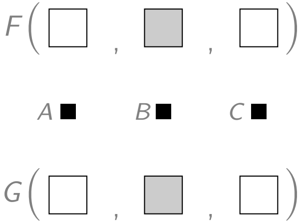

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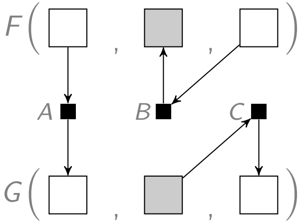

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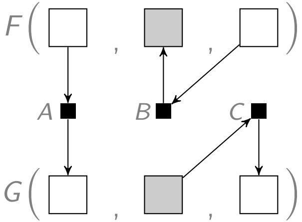


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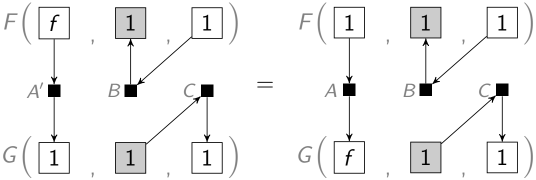

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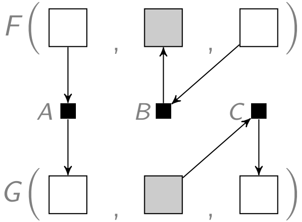


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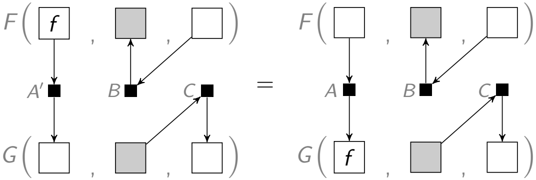

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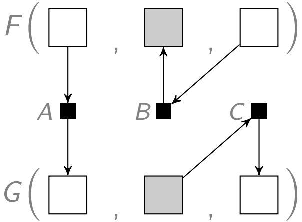


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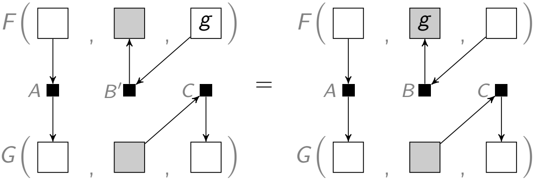

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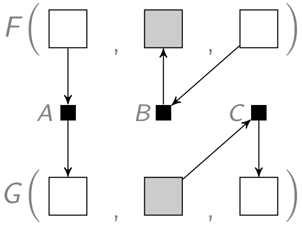


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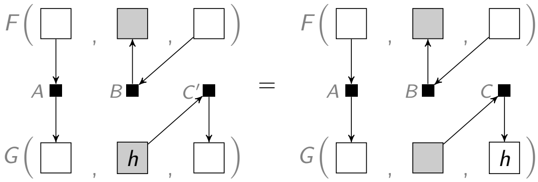

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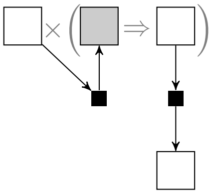

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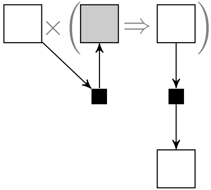


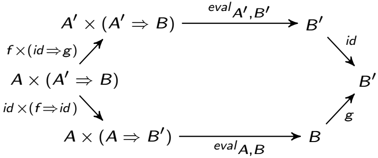


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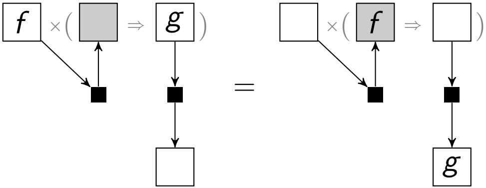
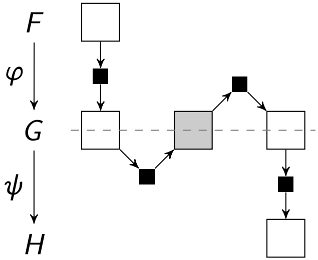


# Eilenberg and Kelly's theorem



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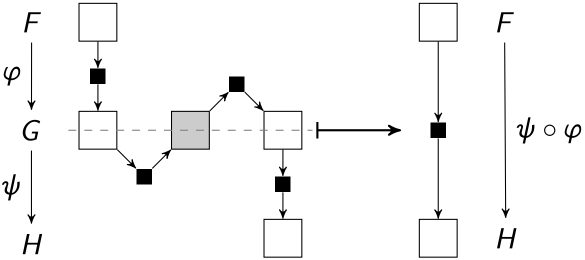

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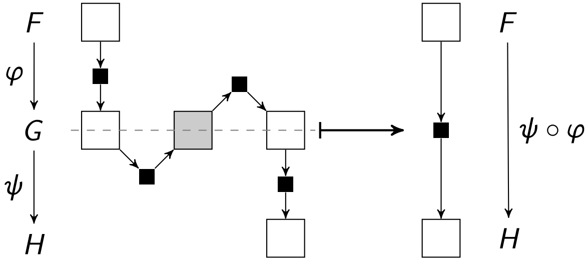
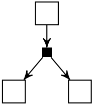
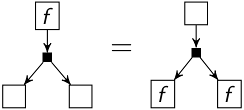



**Theorem (Eilenberg, Kelly 1966)**

*If the composite graph of $\varphi$ and $\psi$ is acyclic, then $\psi \circ \varphi$ is extranatural.*

# Ramifications in the graphs[*]

$\mathbb{C}$ cartesian closed. $(\delta_A \colon A \to A \times A)_{A \in \mathbb{C}}$ is a natural transformation

$\delta \colon id_{\mathbb{C}} \to \times$ *with graph*  . Naturality of $\delta$: 


[*] Cf. Kelly, Many-Variable Functorial Calculus I, 1972.

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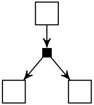
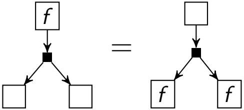
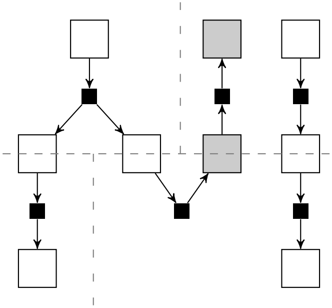
$\delta\colon id_{\mathbb{C}} \to \times$ *with graph*  . Naturality of $\delta$: 


Consider $\chi_{A,B} = (\delta_A \times id_{A \Rightarrow B})\,;(id_A \times eval_{A,B}) : A \times (A \Rightarrow B) \to A \times B$.



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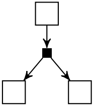
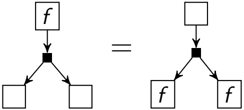
$\delta \colon id_{\mathbb{C}} \to \times$ *with graph*  . Naturality of $\delta$: 



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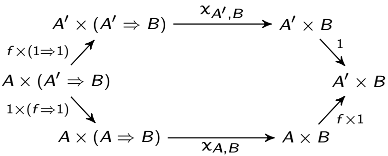



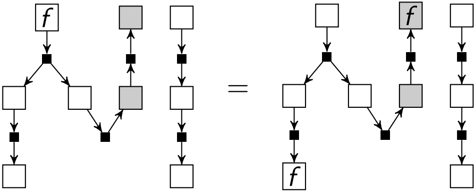



$\chi$ is natural in $B$ and dinatural in $A$.

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# The result

$F\colon \mathbb{C}^\alpha \to \mathbb{D}$, $G\colon \mathbb{C}^\beta \to \mathbb{D}$ functors, where $\alpha, \beta \in \mathsf{List}\{+,-\}$,
$\varphi = (\varphi_{A_1,\dots,A_k})_{A_1,\dots,A_k \in \mathbb{C}}\colon F \to G$ and $\psi = (\psi_{B_1,\dots,B_l})_{B_1,\dots,B_l \in \mathbb{C}}\colon G \to H$ dinatural transformations with graph $\Gamma(\varphi)$ and $\Gamma(\psi)$.

**Theorem**

If the composition of $\Gamma(\varphi)$ and $\Gamma(\psi)$ is acyclic, then $\psi \circ \varphi$ is again dinatural.

# The incidence matrix

Say $n =$ number of upper and lower boxes in $\Gamma(\varphi)$,
$m =$ number of black squares in $\Gamma(\varphi)$.
The *incidence matrix* of $\varphi$ is the $n \times m$ matrix $A$ where

$$A_{i,j} = \begin{cases} -1 & \text{there is an arc from } i \text{ to } j \\ 1 & \text{there is an arc from } j \text{ to } i \\ 0 & \text{otherwise} \end{cases}$$

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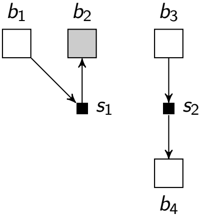


$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \end{array}$$

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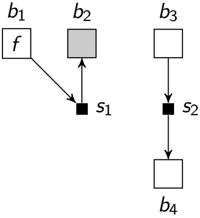


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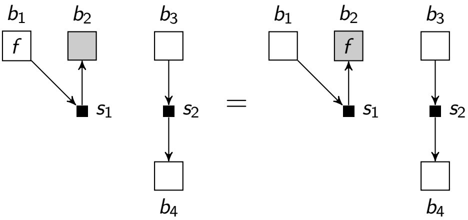
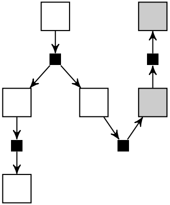



$$\begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

# A reachability problem



$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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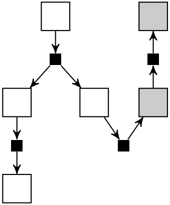

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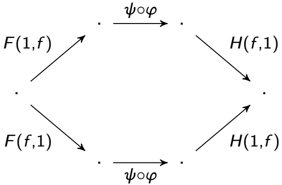


$$M_o(b) = \begin{cases} 1 & b \text{ is a white upper/grey lower box} \\ 0 & \text{otherwise} \end{cases}$$

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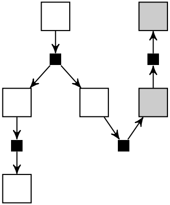

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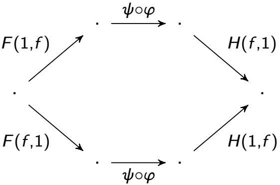


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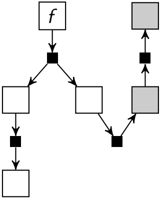
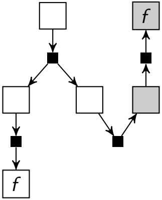
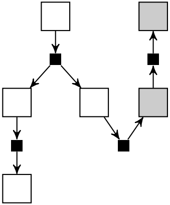
$M_o =$ 

, $M_d =$ 



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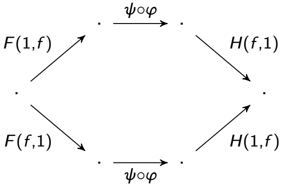


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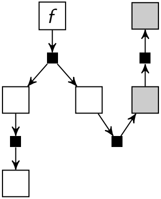


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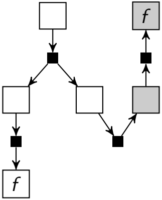


If $M_d$ is reachable from $M_o$, then $\psi \circ \varphi$ is dinatural.

# A reachability result

**Theorem (Ichikawa-Hiraishi 1988, paraphrased)**

*Suppose* $\Gamma(\psi) \circ \Gamma(\varphi)$ *is acyclic and let* $M$, $M'$ *be two markings. Then* $M'$ *is reachable from* $M$ *if and only if there is a non-negative integer solution* $x$ *for*

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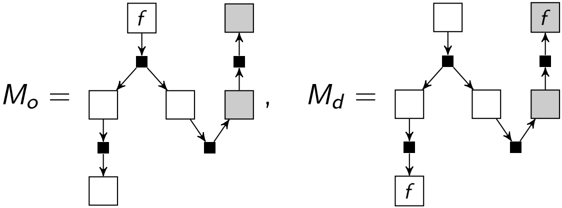

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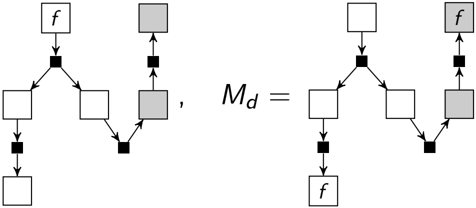


Take $x = [1, \ldots, 1]$, that is, apply the dinaturality condition of $\varphi$ and $\psi$ in each of their variables exactly once: it works no matter how many boxes and squares we have!

# A generalised functor category

**Theorem**

*Let $\varphi\colon F \to G$ and $\psi\colon G \to H$ be dinatural transformations. If their composite graph is acyclic, then $\psi \circ \varphi$ is still dinatural.*

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The category $\{\mathbb{C}, \mathbb{D}\}$ consists of the following data.

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- $\varphi = (\varphi_{A_1,\dots,A_n})\colon F \to G$ is a transformation,
- $\Delta\colon \{1, \dots, n\} \to \{0, 1\}$ is the *discriminant function* such that $\Delta(i) = 1$ implies $\varphi$ dinatural in its $i$-th variable,

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- $\mathcal{G}$ is a graph and can be either:
    - the Eilenberg-Kelly graph of $\varphi$ as defined earlier,
    - a composite of EK graphs of consecutive transformations $\varphi_1, \dots, \varphi_k$, in which case $\varphi = \varphi_k \circ \cdots \circ \varphi_1$.

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