Descent and Monadicity

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Category Theory - CT 2019, University of Edinburgh

Aim

Work

The results are within the general context of 2-category theory, or the so called *formal category theory*.

- [Lucatelli Nunes 2019]
  Semantic Factorization and Descent
- [Lucatelli Nunes 2019]
  Descent data and absolute Kan extensions

Aim

Work

The results are within the general context of 2-category theory.

Talk

I shall sacrifice generality (even in classical results), in order to give an idea of the elementary consequences on the relation between *(classical/Grothendieck) descent theory and monadicity* in the particular case of the 2-category **Cat** (and, more particularly, right adjoint functors). For instance, within the context of:

[Bénabou and Roubaud 1970]
Monades et descente

[Janelidze and Tholen 1994]
Facets of Descent, I

Outline

The category Δ_3

Definition of Δ_3

We denote by Δ_3 the category generated by the diagram

$$\mathbf{1} \underset{d^1}{\overset{d^0}{\rightrightarrows}} \mathbf{2} \quad (s^0 : \mathbf{2} \to \mathbf{1}), \qquad \mathbf{2} \xrightarrow{D^0,\ D^1,\ D^2} \mathbf{3}$$

with the usual (co)simplicial identities

$$D^1 d^0 = D^0 d^0, \quad D^2 d^1 = D^1 d^1, \quad D^2 d^0 = D^0 d^1$$

$$s^0 d^0 = s^0 d^1 = \mathrm{id}_{\mathbf{1}}$$

The descent category

$$\mathcal{A} : \Delta_{\mathbf{3}} \to \mathbf{Cat}$$

$$\mathcal{A}(\mathbf{1}) \underset{\mathcal{A}(d^1)}{\overset{\mathcal{A}(d^0)}{\underset{\xleftarrow{\mathcal{A}(s^0)}}{\rightrightarrows}}} \mathcal{A}(\mathbf{2}) \underset{\mathcal{A}(D^2)}{\overset{\mathcal{A}(D^0)}{\underset{\xrightarrow{\mathcal{A}(D^1)}}{\rightrightarrows}}} \mathcal{A}(\mathbf{3})$$

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Desc $(\mathcal{A})$

- Obj: (X, b); $X \in \mathcal{A}(\mathbf{1})$, $b : \mathcal{A}(d^1)X \to \mathcal{A}(d^0)X$ in $\mathcal{A}(\mathbf{2})$

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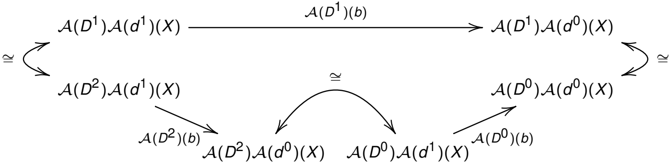
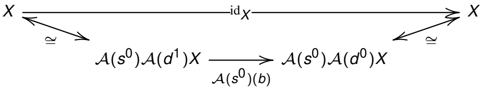
Associativity equation/diagram:



Identity equation/diagram:



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- Obj: (X, b); $X \in \mathcal{A}(\mathbf{1})$, $b : \mathcal{A}(d^1)X \to \mathcal{A}(d^0)X$ in $\mathcal{A}(\mathbf{2})$ satisfying the associativity and identity equations;
- Mor: $\tilde{f} : (X, b) \to (X', b')$ is a morphism $f : X \to X'$ of $\mathcal{A}(\mathbf{1})$ s.t.

$$\mathcal{A}(d^0)(f) \cdot b = b' \cdot \mathcal{A}(d^1)(f).$$

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Functor that forgets the descent data w.r.t. $\mathcal{A}$

$$\mathrm{Desc}\,(\mathcal{A}) \to \mathcal{A}(\mathbf{1})$$

Descent category as a two dimensional limit

$\mathrm{Desc}(\mathcal{A})$ is a two dimensional limit of $\mathcal{A} : \Delta_3 \to \mathbf{Cat}$ in **Cat**.

[Ross Street 1976]
Limits indexed by category-valued 2-functors

[Lucatelli Nunes 2019]
Semantic Factorization and descent

Descent category as a two dimensional limit

$\mathrm{Desc}(\mathcal{A})$ is a two dimensional limit of $\mathcal{A} : \Delta_3 \to$ **Cat** in **Cat**.

The universal property of the descent category

$$\mathbb{D} \xrightarrow{F} \mathcal{A}(\mathbf{1})$$

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$$\left(\begin{array}{ccccc} & & \mathcal{A}(\mathbf{1}) & & \\ & {\scriptstyle F}\nearrow & & \searrow{\scriptstyle \mathcal{A}(d^0)} & \\ \mathbb{D} & & \Uparrow \gamma & & \mathcal{A}(\mathbf{2}) \\ & {\scriptstyle F}\searrow & & \nearrow{\scriptstyle \mathcal{A}(d^1)} & \\ & & \mathcal{A}(\mathbf{1}) & & \end{array} \right) \mapsto \left(\begin{array}{ccc} \mathbb{D} & \xrightarrow{F} & \mathcal{A}(\mathbf{1}) \\ & {\scriptstyle \mathcal{K}^\gamma}\searrow & \nearrow \\ & \mathrm{Desc}\,(\mathcal{A}) & \end{array} \right)$$

$\uparrow$

satisfying associativity and identity equations w.r.t. $\mathcal{A}$.

$$\mathcal{K}^\gamma(W) = (F(Y), \gamma_W)$$

Basic factorization

1. $\mathbb{C}$ with pullbacks;
2. $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to$ **Cat**;

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$$E \times_B E \times_B E \rightrightarrows E \times_B E \rightleftarrows E$$

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$$\mathcal{F}(E) \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \mathcal{F}(E \times_B E) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{F}(E \times_B E \times_B E)$$

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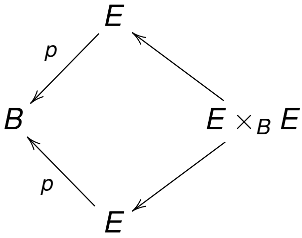

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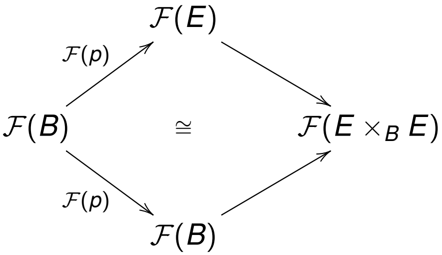

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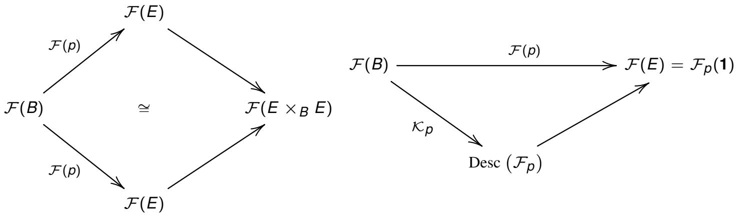

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($\mathcal{F}$-descent factorization)

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($\mathcal{F}$-descent factorization)

Definition

p is of *effective $\mathcal{F}$-descent* if $\mathcal{K}_p$ is an equivalence.

Bénabou-Roubaud Theorem

Hypotheses of the Bénabou-Roubaud Theorem

1. $\mathbb{C}$ with pullbacks;
2. $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to$ **CAT**;
3. $\mathcal{F}(q)! \dashv \mathcal{F}(q)$, for all q;
4. $\mathcal{F}$ satisfies the so called Beck-Chevalley condition.

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coincides with the Eilenberg-Moore factorization of the right adjoint functor $\mathcal{F}(p)$.

Corollary

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Observation

It characterizes *descent via monadicity*

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Corollary

p of effective $\mathcal{F}$-descent if and only if $\mathcal{F}(p)$ is monadic.

Observation

It characterizes *descent via monadicity*: the problem of descent reduces to the problem of monadicity under the hypothesis of the Beck-Chevalley condition.

Basic (Counter)example

$\mathbf{2}$ is the category $0 \xrightarrow{u} 1$

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Facts

$$\begin{array}{ccccc} \mathcal{F}(0) & & \xrightarrow{\mathcal{F}(u)} & & \mathcal{F}(1) = \mathcal{F}_p(\mathbf{1}) \\ & \searrow^{\mathcal{F}(u)} & & \nearrow_{\mathrm{id}} & \\ & & \mathcal{F}(1) \simeq \mathrm{Desc}\,(\mathcal{F}_u) & & \end{array}$$

Basic (Counter)example

$$\mathcal{F} : \mathsf{2}^{\mathrm{op}} \to \mathbf{Cat}$$

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& \mathcal{F}(1) \simeq \mathrm{Desc}\left(\mathcal{F}_u\right) &
\end{array}$$

- u of effective $\mathcal{F}$-descent $\iff$ $\mathcal{F}(u)$ equivalence;

Basic (Counter)example

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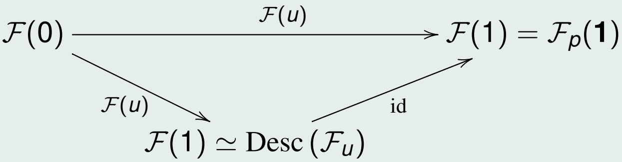
Facts



- u of effective $\mathcal{F}$-descent $\iff$ $\mathcal{F}(u)$ equivalence;

Therefore if G is monadic (and not an equivalence), defining $\mathcal{F}(u) = G$, u is not of effective $\mathcal{F}$-descent but $\mathcal{F}(u)$ is monadic.

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- u of effective $\mathcal{F}$-descent $\iff \mathcal{F}(u)$ equivalence;
- $\mathcal{F}$ satisfies Beck-Chevalley $\iff \mathcal{F}(u)! \dashv \mathcal{F}(u)$ and $\mathcal{F}(u)!$ fully faithful.

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More structured examples

Non-effective of descent morphisms inducing monadic functors

[Manuela Sobral 2004]
Descent for discrete (co)fibrations.

[Margarida Melo 2004]
Master's thesis: Monadicidade e descida - da fibração básica à fibração dos pontos.

Higher cokernel

$$G : \mathbb{A} \to \mathbb{B}$$

Higher cokernel

The higher cokernel $\mathcal{H}_G : \Delta_3 \to \textbf{Cat}$ of $G : \mathbb{A} \to \mathbb{B}$.

[Ross Street 2004]
Categorical and combinatorial aspects of descent theory.

[Steve Lack 2002]
Codescent objects and coherence

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Opcomma category

$$\mathbb{B} \uparrow_G \mathbb{B}$$

[Ross Street 1974]
Elementary cosmoi. I.

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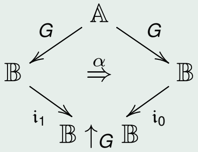

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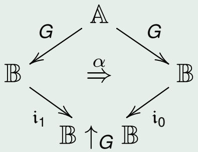

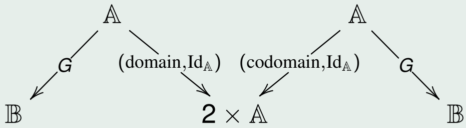

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$$\begin{array}{ccccccc} & & \mathbb{A} & & \mathbb{A} & & \\ & \swarrow^{G} & & \searrow^{(\mathrm{domain},\mathrm{Id}_{\mathbb{A}})} \quad {}^{(\mathrm{codomain},\mathrm{Id}_{\mathbb{A}})}\swarrow & & \searrow^{G} & \\ \mathbb{B} & & & 2 \times \mathbb{A} & & & \mathbb{B} \end{array}$$

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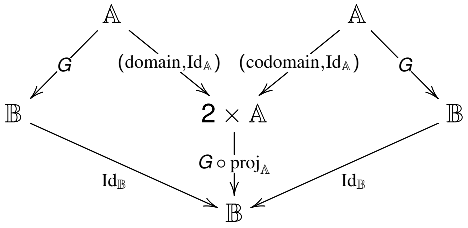

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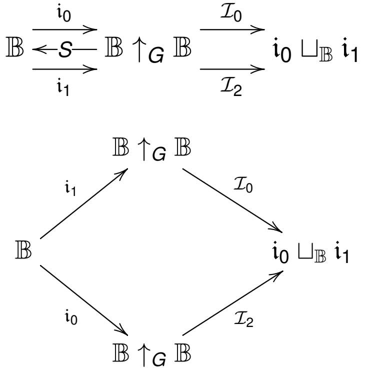

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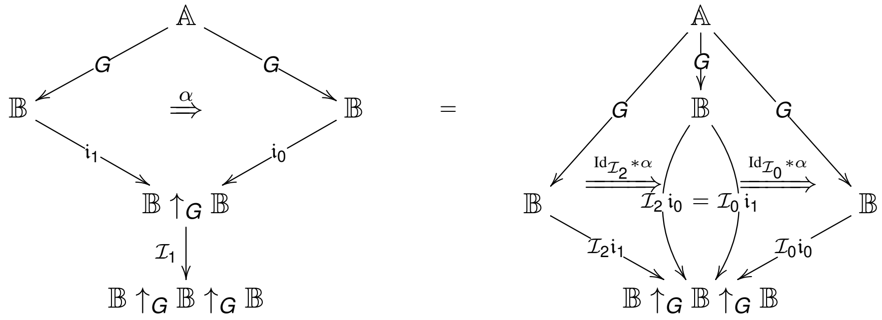
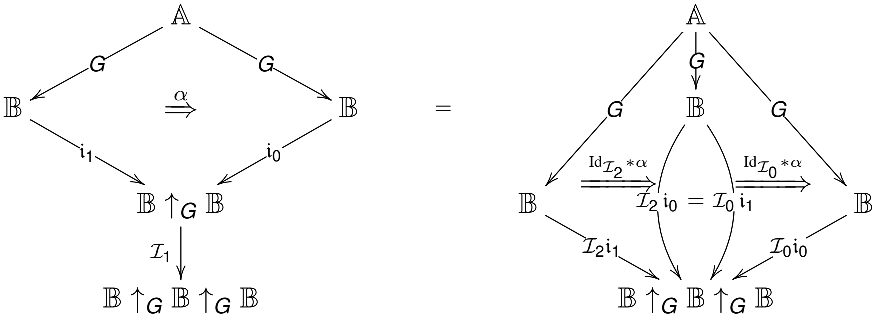


(Descent) factorization of functors

$$\mathcal{H}_G : \mathbf{\Delta_3} \to \mathbf{Cat}$$

$$\mathbb{B} \underset{\mathrm{i}_1}{\overset{\mathrm{i}_0}{\underset{\longleftarrow S \longrightarrow}{\longrightarrow}}} \mathbb{B} \uparrow_G \mathbb{B} \underset{\mathcal{I}_2}{\overset{\mathcal{I}_0}{\underset{\longrightarrow \mathcal{I}_1 \longrightarrow}{\longrightarrow}}} \mathbb{B} \uparrow_G \mathbb{B} \uparrow_G \mathbb{B}$$



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$$\begin{array}{ccccc} & & \mathbb{A} & & \\ & \swarrow^{G} & & \searrow^{G} & \\ \mathbb{B} & & \overset{\alpha}{\Rightarrow} & & \mathbb{B} \\ & \searrow_{\mathrm{i}_1} & & \swarrow_{\mathrm{i}_0} & \\ & & \mathbb{B} \uparrow_G \mathbb{B} & & \end{array}$$

satisfies the associativ. and ident. eq's w.r.t. $\mathcal{H}_G$.

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$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\quad G \quad} & \mathbb{B} = \mathcal{H}_G(\mathbf{1}) \\ & {\scriptstyle \mathcal{K}^{\alpha}}\searrow & \nearrow \\ & & \mathrm{Desc}\,(\mathcal{H}_G) \end{array}$$

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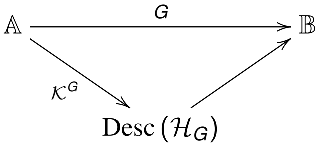
$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\quad G \quad} & \mathbb{B} \\ & {\scriptstyle \mathcal{K}^G := \mathcal{K}^\alpha} \searrow \quad \nearrow & \\ & \mathrm{Desc}\,(\mathcal{H}_G) & \end{array}$$

(factorization (of any functor) induced by the higher cokernel)

Contributions on monadicity via descent

$$G : \mathbb{A} \to \mathbb{B}$$

$$\mathcal{H}_G : \mathbf{\Delta_3} \to \mathbf{Cat}$$



(factorization of G induced by the higher cokernel $\mathcal{H}_G$)

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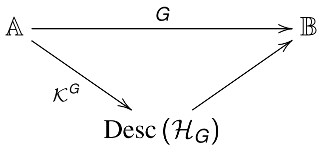
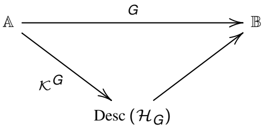


(factorization of G induced by the higher cokernel $\mathcal{H}_G$)

Theorem A

- If G has a left adjoint, then the factorization above coincides with the Eilenberg-Moore factorization of G;
- If G has a right adjoint, then the factorization above coincides with the factorization of G through the coalgebras.

Contributions on monadicity via descent



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If G has a left adjoint: then G is monadic if and only if $\mathcal{K}^G$ is an equivalence (G is effective faithful functor).

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Theorem B

For any pseudofunctor $\mathcal{A} : \Delta_3 \to \mathbf{Cat}$,

$$\mathrm{Desc}\,(\mathcal{A}) \to \mathcal{A}(\mathbf{1})$$

creates absolute limits and colimits.

Contributions on monadicity via descent

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If G has a left adjoint, then the factorization above coincides with the Eilenberg-Moore factorization of G;

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Theorem B

If G is the composition of a functor that forgets descent data w.r.t. some $\mathcal{A} : \Delta_\mathbf{3} \to \mathbf{Cat}$ with any equivalence, then it creates absolute limits and colimits.

Contributions on monadicity via descent

Theorem A

If G has a left adjoint, then the (descent) factorization induced by the higher cokernel coincides with the Eilenberg-Moore factorization of G.

Corollary A.1

If G has a left adjoint: then G is monadic if and only if $\mathcal{K}^G$ is an equivalence (G is effective faithful functor).

Theorem B

If G is the composition of a functor that forgets descent data with any equivalence, then it creates absolute limits and colimits.

Corollary B.1

A right adjoint functor G is monadic if and only if it is a functor that forgets descent data (w.r.t. some $\mathcal{A}$).

Final observation

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Final observation

Corollary B.1

A right adjoint functor G is monadic if and only if it is a functor that forgets descent data (w.r.t. some $\mathcal{A}$).

1. $\mathbb{C}$ with pullbacks;
2. $\mathcal{F} : \mathbb{C}^{\mathrm{op}} \to \mathbf{CAT}$.

Corollary B.1.1

If p is of effective $\mathcal{F}$-descent and $\mathcal{F}(p)$ has a left adjoint, then $\mathcal{F}(p)$ is monadic.

Thank you!