The Constructive Kan–Quillen Model Structure

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The classical Kan–Quillen model structure

Theorem

The category of simplicial sets carries a proper cartesian model structure where

- *weak equivalences are the weak homotopy equivalences,*
- *cofibrations are the monomorphisms,*
- *fibrations are the Kan fibrations.* □

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A constructive version of the model structure would be useful in

- study of models of Homotopy Type Theory;
- understanding homotopy theory of simplicial sheaves.

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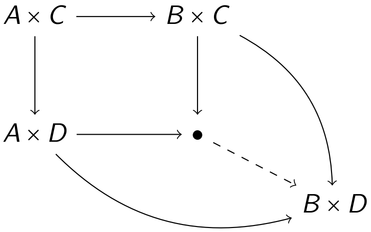
Proofs:

- S. Henry, *A constructive account of the Kan-Quillen model structure and of Kan's* Ex^∞ *functor*
- N. Gambino, C. Sattler, K. Szumiło, *The Constructive Kan–Quillen Model Structure: Two New Proofs*

Fibrations and cofibrations

If $A \to B$ and $C \to D$ are cofibrations, then so is their *pushout product*.
If one of the is trivial, then so is the pushout product.



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Fibration category of Kan complexes

Theorem

The category of Kan complexes is a fibration category, i.e.

- *It has a terminal object and all objects are fibrant.*
- *Pullbacks along fibrations exist and (acyclic) fibrations are stable under pullback.*
- *Every morphism factors as a weak equivalence followed by a fibration.*
- *Weak equivalences satisfy the 2-out-of-6 property.*
- *It has products and (acyclic) fibrations are stable under products.*
- *It has limits of towers of fibrations and (acyclic) fibrations are stable under such limits.*

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use the pushout product property to strictify inverses to acyclic fibrations and show that they are trivial.

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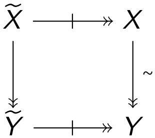

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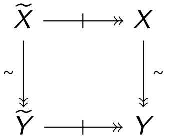

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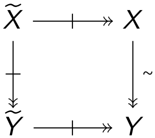

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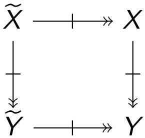

Cofibration category of cofibrant simplicial sets

Theorem

The category of cofibrant simplicial sets is a fibration category, i.e.

- *It has an initial object and all objects are cofibrant.*
- *Pushouts along cofibrations exist and (acyclic) cofibrations are stable under pushout.*
- *Every morphism factors as a cofibration followed by a weak equivalence.*
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- *It has coproducts and (acyclic) cofibrations are stable under coproducts.*
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Dualise by applying $(-)^K$ for all Kan complexes K.

Diagonals of bisimplicial sets

Proposition

If $X \to Y$ is a map between cofibrant bisimiplicial sets such that $X_k \to Y_k$ is a weak homotopy equivalence for all k, then the induced map $\operatorname{diag} X \to \operatorname{diag} Y$ is also a weak homotopy equivalence.

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$$\begin{array}{ccccc} L_kX \times \Delta[k] \cup X_k \times \partial\Delta[k] & & \longrightarrow & & \operatorname{diag}\operatorname{Sk}^{k-1} X \\ & \searrow & & & & \searrow \\ & & X_k \times \Delta[k] & & \longrightarrow & & \operatorname{diag}\operatorname{Sk}^{k} X \end{array}$$

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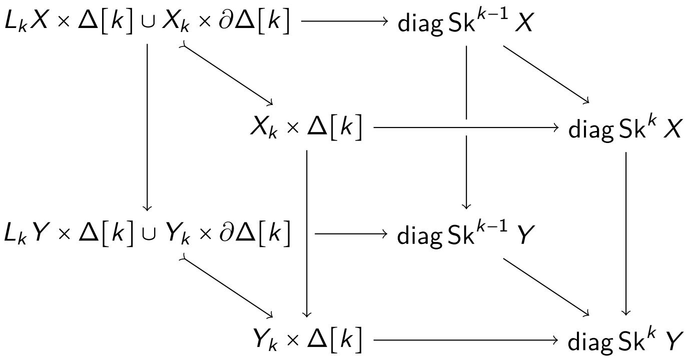

Kan's Ex^∞ functor

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Proposition

- Ex^∞ *preserves finite limits.*
- Ex^∞ *preserves Kan fibrations between cofibrant objects.*
- *If X is cofibrant, then $\mathrm{Ex}^\infty X$ is a Kan complex.*
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The last statement is proven by argument of Latch–Thomason–Wilson.

Kan's Ex^∞ functor

$$
\begin{array}{ccc}
\mathrm{sSet}(\Delta[m] \times \Delta[0], X) & \longrightarrow & \mathrm{sSet}(\Delta[m] \times \Delta[n], X) \\
\downarrow & & \downarrow \\
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\bullet & \longrightarrow & X^{\Delta[m]} \\
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\qquad
\begin{array}{ccc}
X^{\Delta[0]} & \xrightarrow{\simeq} & X^{\Delta[n]} \\
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\qquad
\begin{array}{ccc}
X & \xrightarrow{\sim} & \bullet \\
\downarrow & & \downarrow{\scriptstyle \sim} \\
\mathrm{Ex}\,X & \xrightarrow[\sim]{} & \bullet
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Trivial fibrations vs. acyclic fibrations

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If p is trivial, then it is acyclic – fairly easy.

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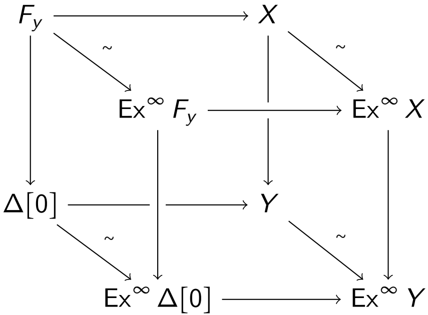
If p is acyclic and X and Y are cofibrant, use Ex^∞:



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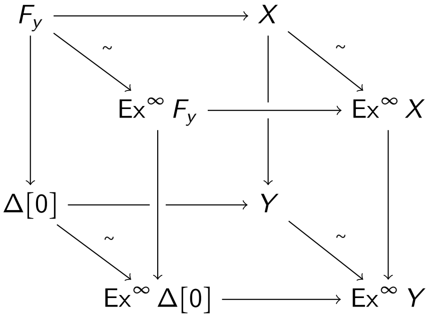

For general X and Y, use the cancellation trick.