Relative Partial Combinatory Algebras over Heyting Categories

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Category Theory, 8 July 2019

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Partial Combinatory Algebras

Definition

A *partial combinatory algebra* (PCA) is a nonempty set A with a *partial* binary operation $A \times A \rightharpoonup A$: $(a, b) \mapsto ab$ for which there exist $\mathsf{k}, \mathsf{s} \in A$ such that:

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(i) $\mathsf{k}ab = a$; (here $abc = (ab)c$)

(ii) $\mathsf{s}ab$ is always defined;

(iii) if $ac(bc)$ is defined, then $\mathsf{s}abc$ is defined and equal to $ac(bc)$.

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Property

If $t(\vec{x}, y)$ is a term, then there exists an $r \in A$ such that:

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Relative PCAs

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A *relative PCA* is a pair (A, C) where A is a PCA, and $C \subseteq A$ closed under the application from A, such that there exist $\mathsf{k}, \mathsf{s} \in C$ witnessing the fact that A is a PCA.

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We view the elements of C as *computable* elements acting on possibly *non-computable* data.

Examples of PCAs

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Kleene's first model $\mathcal{K}_1$ is $\mathbb{N}$ with $mn = \varphi_m(n)$.

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$(\mathcal{P}\mathbb{N}, (\mathcal{P}\mathbb{N})_{\text{r.e.}})$ is a relative PCA.

Assemblies

Definition

The category $\mathrm{Asm}(A, C)$:

(i) has as objects pairs $X = (|X|, E_X)$, where $|X|$ is a set and $E_X \subseteq |X| \times A$ satisfies: for all $x \in |X|$, there is an $a \in A$ with $E_X(x, a)$.

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The category $\mathrm{Asm}(A)$ is a quasitopos.

Question

What does a category of the form $\mathrm{Asm}(A)/I$ or $\mathrm{Asm}(A, C)/I$ look like?

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There is an adjunction

$$\mathsf{Set} \underset{\nabla}{\overset{\Gamma}{\leftrightarrows}} \mathsf{Asm}(A, C)$$

with $\Gamma \dashv \nabla$.

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There is also a notion of *morphism* between HPCAs over $\mathcal{H}$.

The Category HPCA

Proposition (Z)

If (A, ϕ) is an HPCA over $\mathcal{H}$ and $p\colon \mathcal{H} \to \mathcal{G}$ is a Heyting functor, then

$$p^*(A, \phi) := (p(A), \langle p(\phi) \rangle)$$

is an HPCA over $\mathcal{G}$;

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The category HPCA has small products.

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The category $\mathsf{Asm}(A, \phi)$:

(i) has as objects pairs $X = (|X|, E_X)$, where $|X| \in \mathcal{H}$ and $E_X \subseteq |X| \times A$ is such that $\forall x \in |X| \exists a \in A(E_X(x, a))$ is valid in $\mathcal{H}$.

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An applicative morphism $(p, f)\colon (A, \phi) \to (B, \psi)$ also induces a functor $\mathrm{Asm}(p, f)\colon \mathrm{Asm}(A, \phi) \to \mathrm{Asm}(B, \psi)$.

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The functor Asm preserves small products.

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Description of Slice Categories

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Theorem (Z)

$\mathrm{Asm}(A, \phi)/I$ is equivalent to $\mathrm{Asm}((A, \phi)/I)$, where

$$(A, \phi)/I := (|I|^*(A), \langle |I|^*(\phi) \cup \{E_I\} \rangle).$$

(Observe that $E_I \subseteq |I| \times A = |I|^*(A)$.)

Examples of Slice Categories I

Example

Consider $1+1 \in \mathsf{Asm}(\mathcal{K}_1)$. Then $\mathcal{K}_1/1+1 \cong ((\mathcal{K}_1, \mathcal{K}_1), \phi_{\mathsf{max}})$, so

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An arrow $(f_0, f_1)\colon (X_0, X_1) \to (Y_0, Y_1)$ of $\mathsf{Asm}(\mathcal{K}_1)^2$ belongs to $\mathsf{Asm}((\mathcal{K}_1, \mathcal{K}_1), \phi)$ if f_0 and f_1 have a *simultaneous* tracker.

Examples of Slice Categories II

Example

Consider $\Sigma \in \mathrm{Asm}(\mathcal{K}_1)$ where $|\Sigma| = 2$ and

$$E_\Sigma = \{(0, n) \mid nn\downarrow\} \cup \{(1, n) \mid nn\uparrow\}.$$

Then $\mathcal{K}_1/\Sigma = ((\mathcal{K}_1, \mathcal{K}_1), \phi)$ is *not* generated by some $C \subseteq (\mathbb{N}, \mathbb{N})$.

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Examples of Slice Categories III

Example

The natural numbers object $N \in \mathrm{Asm}(\mathcal{K}_1)$ is given by $|N| = \mathbb{N}$ and $E_N = \delta \subseteq \mathbb{N} \times \mathbb{N}$. We have $\mathcal{K}_1/N \cong ((\mathcal{K}_1)_{n\in\mathbb{N}}, \phi)$, where

$$\phi = \{(U_n)_{n\in\mathbb{N}} \mid \exists a \in \mathbb{N}^{\mathbb{N}}_{\mathrm{rec}} \forall n \in \mathbb{N}(a_n \in U_n)\}.$$

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so $\mathrm{Asm}(\mathcal{P}\mathbb{N})/N \simeq \prod_{n\in\mathbb{N}} \mathrm{Asm}(\mathcal{P}\mathbb{N})$.

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Geometric morphisms

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\mathrm{Asm}(B,\psi) \arrow[d, shift right, "\Gamma"'] & & \mathrm{Asm}(A,\phi) \arrow[ll, "\mathrm{Asm}(p^*, f)"'] \arrow[d, shift right, "\Gamma"'] \\
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Question

When doen $\mathrm{Asm}(p^*, f)$ have a right adjoint?

Computational Density

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Example

If $I \in \mathrm{Asm}(A, \phi)$, then there is a canonical applicative morphism $(A, \phi) \to (A, \phi)/I$, which is computationally dense.

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