Coherence modulo and double groupoids

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joint work with Philippe Malbos

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Plan

I. Introduction and motivations

Motivations: algebraic context

- Algebraic rewriting: constructive methods to study algebraic structures presented by generators and relations.

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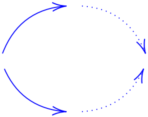

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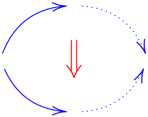

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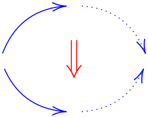

- If a group $G = \langle X \mid R \rangle$ is presented as a monoid $M = \langle X \coprod \overline{X} \mid R \cup \{xx^- \xrightarrow{\alpha_x} 1, x^- x \xrightarrow{\overline{\alpha_x}} 1\}$,

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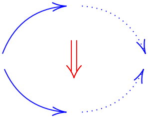

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$$\begin{tikzcd}
 & x \arrow[dd, bend left, "\parallel"] \\
xx^{-1}x \arrow[ur, bend left, "\alpha_x x"] \arrow[dr, bend right, "x\overline{\alpha_x}"'] & \\
 & x
\end{tikzcd}$$

is an artefact induced by the algebraic structure and should not be considered as a syzygy.

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  Example: Isotopy relations

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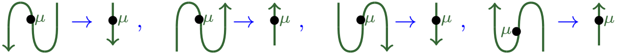
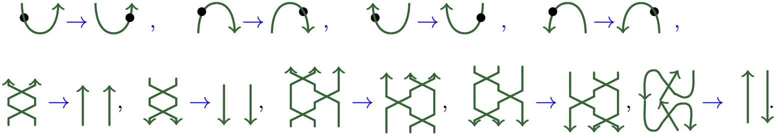


for μ in $\{0, 1\}$



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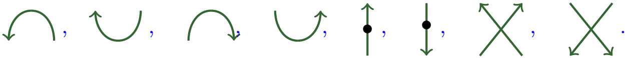

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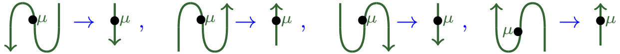

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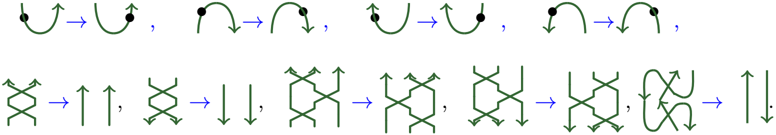

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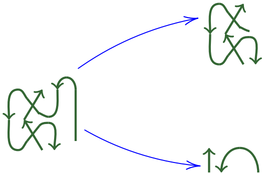

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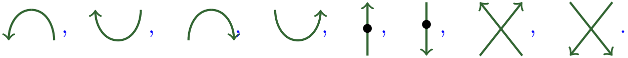

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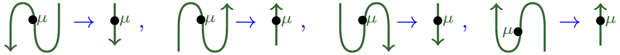

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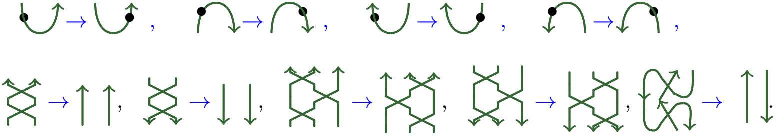

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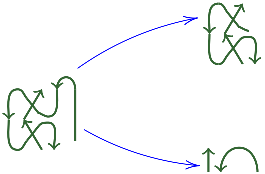

Not confluent !

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$$\begin{tikzcd} u \arrow[r, "R"] \arrow[d, "E"'] & u' \arrow[r, dotted, "R"] & w \arrow[d, dotted, "E"] \\ v \arrow[r, "R"'] & v' \arrow[r, dotted, "R"'] & w' \end{tikzcd}$$

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  - Rewriting with any system S such that $R \subseteq S \subseteq {}_ER_E$, Jouannaud - Kirchner '84.
- Main interest and results for ${}_ER$.

$$\begin{tikzcd} u \arrow[r, dotted, "{}_ER"] \arrow[d, "E"'] & v \arrow[d, "||"] \\ u' \arrow[r, "R"'] & v \end{tikzcd}$$

II. Double groupoids

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- It gives four related categories

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where $\mathbf{C}^{sh}$ is the category $\mathbf{C}_1$ and $\mathbf{C}^{vo}$ is the category $\mathbf{C}_0$.

- Elements of $\mathbf{C}^o$: point cells, elements of $\mathbf{C}^h$ and $\mathbf{C}^v$: horizontal cells and vertical cells.

$$x_1 \xrightarrow{f} x_2 \qquad\qquad \begin{tikzcd} x_1 \arrow[d, "e"'] \\ x_2 \end{tikzcd}$$

Double groupoids

- Elements of $\mathbf{C}_s$ are square cells:

$$\begin{tikzcd} \cdot \arrow[r, "\partial^h_{-,\mathbf{1}}(A)"] \arrow[d, "\partial^v_{-,\mathbf{1}}(A)"'] \arrow[dr, phantom, "\Downarrow A"] & \cdot \arrow[d, "\partial^v_{+,\mathbf{1}}(A)"] \\ \cdot \arrow[r, "\partial^h_{+,\mathbf{1}}(A)"'] & \cdot \end{tikzcd}$$

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- Compositions

$$
\begin{tikzcd}
x_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[r, "f_2"] \arrow[d, "e_2"] \arrow[dr, phantom, "\Downarrow B"] & x_3 \arrow[d, "e_3"] \\
y_1 \arrow[r, "g_1"'] & y_2 \arrow[r, "g_2"'] & y_3
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
x_1 \arrow[r, "f_1 \circ^h f_2"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A \diamond^v B"] & x_3 \arrow[d, "e_3"] \\
y_1 \arrow[r, "g_1 \circ^h g_2"'] & y_3
\end{tikzcd}
$$

for all x_i, y_i, z_i in $\mathbf{C}^o$, f_i in $\mathbf{C}^h$, e_i, e_i' in $\mathbf{C}^v$ and A, A', B in $\mathbf{C}^s$.

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x_1 \arrow[r, "f_1 \circ^h f_2"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A \diamond^v B"] & x_3 \arrow[d, "e_3"] \\
y_1 \arrow[r, "g_1 \circ^h g_2"'] & y_3
\end{tikzcd}
$$

$$
\begin{tikzcd}
x_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[d, "e_2"] \\
y_1 \arrow[r, "f_2"'] \arrow[d, "e'_1"'] \arrow[dr, phantom, "\Downarrow A'"] & y_2 \arrow[d, "e'_2"] \\
z_1 \arrow[r, "f_3"'] & z_2
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
x_1 \arrow[r, "f_1"] \arrow[d, "e_1 \circ^v e'_1"'] \arrow[dr, phantom, "\Downarrow A \diamond^h A'"] & x_2 \arrow[d, "e_2 \circ^v e'_2"] \\
z_1 \arrow[r, "f_3"'] & z_2
\end{tikzcd}
$$

for all x_i, y_i, z_i in $\mathbf{C}^o$, f_i in $\mathbf{C}^h$, e_i, e'_i in $\mathbf{C}^v$ and A, A', B in $\mathbf{C}^s$.

Double groupoids

- These compositions satisfy the middle four interchange law:

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$$\diamond^h$$

$$\begin{tikzcd} y_1 \arrow[r, "g_1"] \arrow[d, "e_1'"'] \arrow[dr, phantom, "\Downarrow A'"] & y_2 \arrow[d, "e_2'"] \\ z_1 \arrow[r, "h_1"] & z_2 \end{tikzcd}$$

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x_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[d, "e_2"] & & x_2 \arrow[r, "f_2"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Downarrow B"] & x_3 \arrow[d, "e_3"] \\
y_1 \arrow[r, "g_1"] & y_2 & & y_2 \arrow[r, "g_2"] & y_3 \\
{} \arrow[r, phantom, "\diamond^h"] & {} & \diamond^v & {} \arrow[r, phantom, "\diamond^h"] & {} \\
y_1 \arrow[r, "g_1"] \arrow[d, "e_1'"'] \arrow[dr, phantom, "\Downarrow A'"] & y_2 \arrow[d, "e_2'"] & & y_2 \arrow[r, "g_2"] \arrow[d, "e_2'"'] \arrow[dr, phantom, "\Downarrow B'"] & y_3 \arrow[d, "e_3'"] \\
z_1 \arrow[r, "h_1"] & z_2 & & z_2 \arrow[r, "h_2"] & z_3
\end{tikzcd}
$$

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$$\left(\begin{array}{c}\begin{tikzcd} x_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[d, "e_2"] \\ y_1 \arrow[r, "g_1"] & y_2 \end{tikzcd} \\ \diamond^h \\ \begin{tikzcd} y_1 \arrow[r, "g_1"] \arrow[d, "e_1'"'] \arrow[dr, phantom, "\Downarrow A'"] & y_2 \arrow[d, "e_2'"] \\ z_1 \arrow[r, "h_1"] & z_2 \end{tikzcd}\end{array}\right) \diamond^v \left(\begin{array}{c}\begin{tikzcd} x_2 \arrow[r, "f_2"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Downarrow B"] & x_3 \arrow[d, "e_3"] \\ y_2 \arrow[r, "g_2"] & y_3 \end{tikzcd} \\ \diamond^h \\ \begin{tikzcd} y_2 \arrow[r, "g_2"] \arrow[d, "e_2'"'] \arrow[dr, phantom, "\Downarrow B'"] & y_3 \arrow[d, "e_3'"] \\ z_2 \arrow[r, "h_2"] & z_3 \end{tikzcd}\end{array}\right) =$$

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\end{tikzcd}
\qquad
\begin{tikzcd}
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y_2 \arrow[r, "g_2"] & y_3
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y_1 \arrow[r, "g_1"] & y_2
\end{tikzcd}
\diamond^v
\begin{tikzcd}
x_2 \arrow[r, "f_2"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Downarrow B"] & x_3 \arrow[d, "e_3"] \\
y_2 \arrow[r, "g_2"] & y_3
\end{tikzcd}
$$

$$\diamond^h \qquad \diamond^v \qquad \diamond^h \qquad =$$

$$
\begin{tikzcd}
y_1 \arrow[r, "g_1"] \arrow[d, "e'_1"'] \arrow[dr, phantom, "\Downarrow A'"] & y_2 \arrow[d, "e'_2"] \\
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y_2 \arrow[r, "g_2"] \arrow[d, "e'_2"'] \arrow[dr, phantom, "\Downarrow B'"] & y_3 \arrow[d, "e'_3"] \\
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$$

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\diamond^h & \diamond^v & \diamond^h & = & & \diamond^h & \\
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$$

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\\
\diamond^h & \diamond^v & \diamond^h
\\
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\end{array}
$$

- Double groupoid: double category in which horizontal, vertical and square cells are invertible.

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$$
\begin{array}{ccccccc}
\begin{tikzcd} x_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[d, "e_2"] \\ y_1 \arrow[r, "g_1"] & y_2 \end{tikzcd}
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\diamond^h & \diamond^v & \diamond^h & = & & \diamond^h & \\
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- n-category enriched in double groupoids: n-category $\mathcal{C}$ such that any homset $\mathcal{C}_n(x, y)$ is a double groupoid.

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\begin{tikzcd}
x_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[d, "e_2"] & & x_2 \arrow[r, "f_2"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Downarrow B"] & x_3 \arrow[d, "e_3"] & & x_1 \arrow[r, "f_1"] \arrow[d, "e_1"'] \arrow[dr, phantom, "\Downarrow A"] & x_2 \arrow[d, "e_2"] & \diamond^v & x_2 \arrow[r, "f_2"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Downarrow B"] & x_3 \arrow[d, "e_3"] \\
y_1 \arrow[r, "g_1"] & y_2 & & y_2 \arrow[r, "g_2"] & y_3 & & y_1 \arrow[r, "g_1"] & y_2 & & y_2 \arrow[r, "g_2"] & y_3 \\
\diamond^h & & \diamond^v & \diamond^h & & = & & & \diamond^h & & \\
y_1 \arrow[r, "g_1"] \arrow[d, "e'_1"'] \arrow[dr, phantom, "\Downarrow A'"] & y_2 \arrow[d, "e'_2"] & & y_2 \arrow[r, "g_2"] \arrow[d, "e'_2"'] \arrow[dr, phantom, "\Downarrow B'"] & y_3 \arrow[d, "e'_3"] & & y_1 \arrow[r, "g_1"] \arrow[d, "e'_1"'] \arrow[dr, phantom, "\Downarrow A'"] & y_2 \arrow[d, "e'_2"] & \diamond^v & y_2 \arrow[r, "g_2"] \arrow[d, "e'_2"'] \arrow[dr, phantom, "\Downarrow B'"] & y_3 \arrow[d, "e'_3"] \\
z_1 \arrow[r, "h_1"] & z_2 & & z_2 \arrow[r, "h_2"] & z_3 & & z_1 \arrow[r, "h_1"] & z_2 & & z_2 \arrow[r, "h_2"] & z_3
\end{tikzcd}
$$

- Double groupoid: double category in which horizontal, vertical and square cells are invertible.

- n-category enriched in double groupoids: n-category $\mathcal{C}$ such that any homset $\mathcal{C}_n(x, y)$ is a double groupoid.

- Horizontal $(n+1)$-category: category of rewritings, vertical $(n+1)$-category: category of modulo rules.

Polygraphs

- Polygraphs are higher-dimensional generating systems of higher-dimensional globular strict categories.

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P_0^* & \\
P_0 \arrow[u, equal] & P_1 \arrow[ul, shift left] \arrow[ul, shift right]
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\end{tikzcd}$$

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$$
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P_0^* & P_1^* & P_2^* \\
P_0 \arrow[u, equal] & P_1 \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] & P_2 \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right]
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P_0^* & P_1^* & P_2^* & (\ldots) & P_{n-1}^* & P_n^* \\
P_0 \arrow[u, equal] & P_1 \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] & P_2 \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] & (\ldots) \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] & P_{n-1} \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] & P_n \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right]
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 & & & & & P_n^\top
\end{tikzcd}
$$

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P_0^* & P_1^* & P_2^* & (\ldots) & P_{n-1}^* & P_n^* \\
P_0 \arrow[u, equal] & P_1 \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] & P_2 \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] & (\ldots) \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] & P_{n-1} \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] & P_n \arrow[u] \arrow[ul, shift left] \arrow[ul, shift right] \arrow[d] \\
 & & & & & P_n^\top
\end{tikzcd}
$$

- An $(n-1)$-category $\mathcal{C}$ is presented by an n-polygraph $(P_0, \ldots, P_n)$ if

$$\mathcal{C} \simeq P_{n-1}^* / \equiv_{P_n}$$

Double $(n+2, n)$-polygraphs

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$$P^v_{n+\mathbf{1}} \rightrightarrows P^*_n \leftleftarrows P^h_{n+\mathbf{1}}$$

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$$
\begin{tikzcd}
(P^v_{n+\mathbf{1}})^* \arrow[drr, shift left] \arrow[drr, shift right] & & & & (P^h_{n+\mathbf{1}})^* \arrow[dll, shift left] \arrow[dll, shift right] \\
P^v_{n+\mathbf{1}} \arrow[u] \arrow[rr, shift left] \arrow[rr, shift right] & & P^*_n & & P^h_{n+\mathbf{1}} \arrow[u] \arrow[ll, shift left] \arrow[ll, shift right]
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  - two $(n+1)$-polygraphs P^v and P^h such that $P^v_k = P^h_k$ for $k \leq n$,
  - a 2-square extension P^s of the pair of $(n+1)$-categories $((P^v)^*, (P^h)^*)$, that is a set equipped with four maps making Γ a 2-cubical set.

$$\begin{tikzcd}
 & & P^s \arrow[dll, shift left] \arrow[dll, shift right] \arrow[drr, shift left] \arrow[drr, shift right] & & \\
(P^v_{\mathbf{n+1}})^* \arrow[drr, shift left] \arrow[drr, shift right] & & & & (P^h_{\mathbf{n+1}})^* \arrow[dll, shift left] \arrow[dll, shift right] \\
P^v_{\mathbf{n+1}} \arrow[u] \arrow[rr, shift left] \arrow[rr, shift right] & & P^*_n & & P^h_{\mathbf{n+1}} \arrow[u] \arrow[ll, shift left] \arrow[ll, shift right]
\end{tikzcd}$$

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$$\begin{tikzcd}
 & P^s \arrow[dl, shift left] \arrow[dl, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & \\
(P^v_{n+\mathbf{1}})^* \arrow[dr, shift left] \arrow[dr, shift right] & & (P^h_{n+\mathbf{1}})^* \arrow[dl, shift left] \arrow[dl, shift right] \\
P^v_{n+\mathbf{1}} \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & P_n^* & P^h_{n+\mathbf{1}} \arrow[u] \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

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$$
\begin{tikzcd}
 & & P^s \arrow[dll, shift left] \arrow[dll, shift right] \arrow[drr, shift left] \arrow[drr, shift right] & & \\
(P^v_{n+1})^* \arrow[drr, shift left] \arrow[drr, shift right] & & & & (P^h_{n+1})^* \arrow[dll, shift left] \arrow[dll, shift right] \\
P^v_{n+1} \arrow[u] \arrow[rr, shift left] \arrow[rr, shift right] & & P^*_n & & P^h_{n+1} \arrow[u] \arrow[ll, shift left] \arrow[ll, shift right]
\end{tikzcd}
$$

- A double $(n+2, n)$-polygraph is a double n-polygraph in which P^s is defined on $((P^v)^\top, (P^h)^\top)$.

- A double $(n+2, n)$-polygraph (P^v, P^h, P^s) generates a free $(n-1)$-category enriched in double groupoids, denoted by $(P^v, P^h, P^s)^{\top\!\!\top}$.

Acyclicity

- A 2-square extension P^s of $((P^v)^\top, (P^h)^\top)$ is acyclic if for any square

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- From Squier's theorem, $(E, \emptyset, \mathrm{Cd}(E))$ is a 2-fold coherent presentation of $\mathbf{C}$.

III. Polygraphs modulo

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  $$\gamma^{{}_E R_E} : {}_E R_E \to \mathrm{Sph}_{n-1}(R^*_{n-1})$$

  where ${}_E R_E$ is the set of triples (e, f, e') in $E^\top \times R^{*(1)} \times E^\top$ such that

$$\begin{tikzcd}[column sep=huge]
u \arrow[r, bend left=80, ""{name=A, below}] \arrow[r, bend left=30, ""{name=B, above}, ""{name=B2, below}] \arrow[r, bend right=30, ""{name=C, above}, ""{name=C2, below}] \arrow[r, bend right=80, ""{name=D, above}] & v
\arrow[Rightarrow, from=A, to=B, "e"]
\arrow[Rightarrow, from=B2, to=C, "f"]
\arrow[Rightarrow, from=C2, to=D, "e'"]
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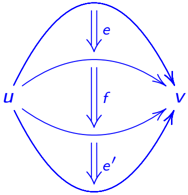

and the map $\gamma^{\,{}_E R_E}$ is defined by $\gamma^{\,{}_E R_E}(e, f, e') = (\partial_{-,n-1}(e), \partial_{+,n-1}(e'))$.

Branchings and confluence modulo

- A branching modulo E of the n-polygraph modulo S is a triple (f, e, g) where f and g are in S_n^* and e is in $E_n^\top$, such that:

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- Confluence modulo E (resp. local confluence modulo E): any branching (resp. local branching) of S modulo E is confluent modulo E.

IV. Coherence modulo

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- We consider Γ a 2-square extension of $(E^{\top}, S^{*})$.
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$$\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e"'] & u' \arrow[r, dotted, "f'"] \arrow[d, phantom, "\Downarrow A"] & w \arrow[d, dotted, "e'"] \\
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$$\begin{tikzcd} u \star_i v \arrow[r, "f \star_i v"] \arrow[d, "u \star_i e"'] & u' \star_i v \arrow[d, "u' \star_i e"] \\ u \star_i v' \arrow[r, "f \star_i v'"'] & u' \star_i v' \end{tikzcd} \qquad \begin{tikzcd} w \star_i u \arrow[r, "w \star_i f"] \arrow[d, "e' \star_i u"'] & w \star_i u' \arrow[d, "e' \star_i u'"] \\ w' \star_i u \arrow[r, "w' \star_i f"'] & w' \star_i u' \end{tikzcd}$$

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  - $E \rtimes \Gamma$ is to avoid "redundant" elements in Γ for different squares corresponding to the same branching of S modulo E:

$$\begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[r, "f'"] & v' \arrow[d, "e'"] \\ u' \arrow[r, "g = e_1 g_1 e_2"'] & w \arrow[r, dotted, "g'"'] & w' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} u \arrow[r, "f"] \arrow[d, "e \star_{n-1} e_1"'] & v \arrow[r, dotted, "f'"] & v' \arrow[d, "e'"] \\ u_1 \arrow[r, "g_1 e_2"'] & w \arrow[r, "g'"'] & w' \end{tikzcd}$$

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b): $$\begin{tikzcd} u \arrow[r, "S^{*(1)}"] \arrow[d, "E^{\top(1)}"'] & v \arrow[r, dotted, "S^*"] \arrow[d, phantom, "\Downarrow B"] & v' \arrow[d, dotted, "E^\top"] \\ u' \arrow[rr, dotted, "S^*"'] & & w \end{tikzcd}$$

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\end{tikzcd}
$$

b):
$$
\begin{tikzcd}
u \arrow[r, "S^{*(1)}"] \arrow[d, "E^{\top(1)}"'] & v \arrow[r, dotted, "S^{*}"] \arrow[d, phantom, "\Downarrow B"] & v' \arrow[d, dotted, "E^{\top}"] \\
u' \arrow[rr, dotted, "S^{*}"'] & & w
\end{tikzcd}
$$

  for any local branching of S modulo E.
  - S satisfies properties **a)** and **b)** for any critical branching of S modulo E.

- For $S = {}_E R$, property **b)** is trivially satisfied.

$$
\begin{tikzcd}
u \arrow[r, "f"] \arrow[d, "e"'] & v \arrow[d, dotted, "\|"] \\
v' \arrow[r, dotted, "e^{-}\cdot f"'] & v
\end{tikzcd}
$$

Coherence modulo

- A set X of $(n-1)$-cells in R^*_{n-1} is E-normalizing with respect to S if for any u in X,

$$\mathrm{NF}(S, u) \cap \mathrm{Irr}(E) \neq \emptyset.$$

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- **Theorem.** [D.-Malbos '18] Let (R, E, S) be an n-polygraph modulo, and Γ be a square extension of $(E^\top, S^*)$ such that
  - E is convergent,
  - S is Γ-confluent modulo E,
  - $\mathrm{Irr}(E)$ is E-normalizing with respect to S,
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  then $E \rtimes \Gamma \cup \mathsf{Peiff}(E,S) \cup \mathrm{Cd}(E)$ is acyclic.

Coherent completion

- Coherent completion modulo E of S: square extension of $(E^\top, S^\top)$ containing square cells $A_{f,g}$ and $B_{f,e}$:

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- **Corollary.** [D.-Malbos '18] Let (R, E, S) be an n-polygraph modulo such that
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Example: diagrammatic rewriting modulo isotopy

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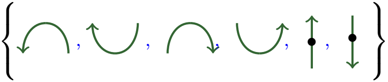
  - $E_2 =$ 

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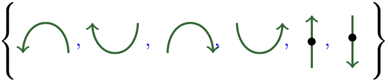
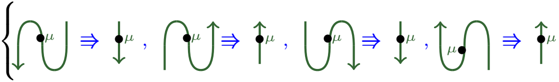
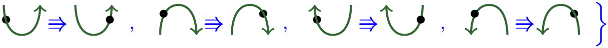
  - $E_2 =$ 
  - $E_3 =$  for μ in $\{0, 1\}$,



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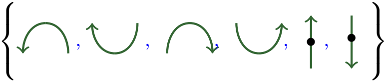
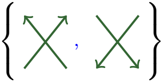
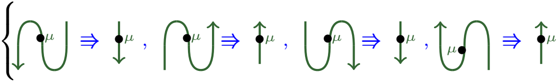
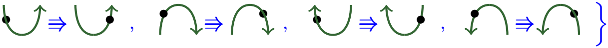
  - $E_2 =$  $R_2 = E_2 \coprod$ 
  - $E_3 =$  for μ in $\{0, 1\}$,



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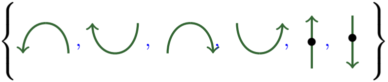
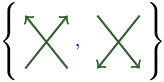
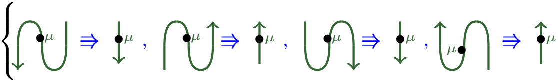
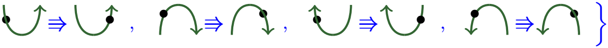
  - $E_2 =$  $R_2 = E_2 \coprod$ 
  - $E_3 =$  for μ in $\{0, 1\}$,
    
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    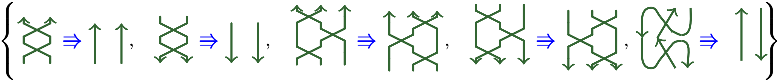

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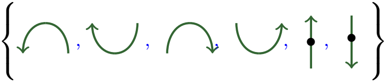
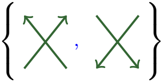
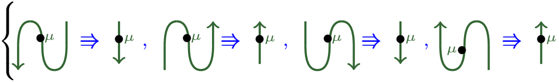
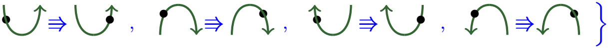
  - $E_2 =$  $R_2 = E_2 \coprod$ 
  - $E_3 =$  for μ in $\{0, 1\}$,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    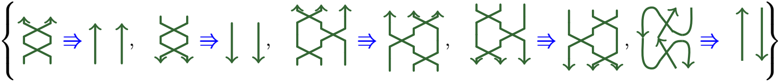
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Thank you !