One: a characterization of skeletal objects for the Aufhebung of Level 0 in certain toposes of spaces

M. Menni

Conicet
and
Universidad Nacional de La Plata

Levels

Let $\mathcal{E}$ be a topos.

Definition

A level l (of $\mathcal{E}$) is a string of adjoints

$$\begin{array}{ccccc} & & \mathcal{E} & & \\ l_! \Big\uparrow & \dashv & \Big\downarrow l^* & \dashv & \Big\uparrow l_* \\ & & \mathcal{E}_l & & \end{array}$$

with fully faithful $l_!, l_* : \mathcal{E}_l \to \mathcal{E}$.

Equivalently, a level is an essential subtopos of $\mathcal{E}$.
($l_* : \mathcal{E}_l \to \mathcal{E}$ is the full subcategory of l-sheaves.)

Levels and ‘dimensions’

Levels and ‘dimensions’

“The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level.”

Lawvere’s *Some thoughts on the future of category theory*
LNM 1488, 1991.

Levels and 'dimensions'

"The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level."

Lawvere's *Some thoughts on the future of category theory*
LNM 1488, 1991.

Let $l_! \dashv l^* \dashv l_*$ be a level of $\mathcal{E}$.

For X in $\mathcal{E}$, the counit $l_!(l^*X) \to X$ is the l-skeleton of X.

So,

Levels and 'dimensions'

"The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level."

Lawvere's *Some thoughts on the future of category theory*
LNM 1488, 1991.

Let $l_! \dashv l^* \dashv l_*$ be a level of $\mathcal{E}$.

For X in $\mathcal{E}$, the counit $l_!(l^*X) \to X$ is the l-skeleton of X.

So, the object X is said to be l-skeletal if its l-skeleton is an iso and

Levels and 'dimensions'

"The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level."

Lawvere's *Some thoughts on the future of category theory*
LNM 1488, 1991.

Let $l_! \dashv l^* \dashv l_*$ be a level of $\mathcal{E}$.

For X in $\mathcal{E}$, the counit $l_!(l^*X) \to X$ is the l-skeleton of X.

So, the object X is said to be l-skeletal if its l-skeleton is an iso and $l_! : \mathcal{E}_l \to \mathcal{E}$ is the full subcat of l-skeletal objects

Intuition:

Levels and 'dimensions'

"The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level."

Lawvere's *Some thoughts on the future of category theory*
LNM 1488, 1991.

Let $l_! \dashv l^* \dashv l_*$ be a level of $\mathcal{E}$.

For X in $\mathcal{E}$, the counit $l_!(l^*X) \to X$ is the l-skeleton of X.

So, the object X is said to be l-skeletal if its l-skeleton is an iso and $l_! : \mathcal{E}_l \to \mathcal{E}$ is the full subcat of l-skeletal objects

Intuition: X is l-skeletal iff $\dim X \leq l$.

Also:

Levels and 'dimensions'

"The basic idea is simply to identify dimensions with levels and then try to determine what the general dimensions are in particular examples. More precisely, a space may be said to have (less than or equal to) the dimension grasped by a given level if it belongs to the negative (left adjoint inclusion) incarnation of that level."

Lawvere's *Some thoughts on the future of category theory* LNM 1488, 1991.

Let $l_! \dashv l^* \dashv l_*$ be a level of $\mathcal{E}$.

For X in $\mathcal{E}$, the counit $l_!(l^*X) \to X$ is the l-skeleton of X.

So, the object X is said to be l-skeletal if its l-skeleton is an iso and $l_! : \mathcal{E}_l \to \mathcal{E}$ is the full subcat of l-skeletal objects

Intuition: X is l-skeletal iff $\dim X \leq l$.

Also: l has monic skeleta if the l-skeleton of every object is monic.

Some simple examples

Some simple examples

Proposition

If $\mathcal{C}$ is a small category then restriction along a full inclusion $\mathcal{B} \to \mathcal{C}$ is the inverse image of a level $\widehat{\mathcal{B}} \to \widehat{\mathcal{C}}$.

Some simple examples

Proposition

If $\mathcal{C}$ is a small category then restriction along a full inclusion $\mathcal{B} \to \mathcal{C}$ is the inverse image of a level $\widehat{\mathcal{B}} \to \widehat{\mathcal{C}}$.

1. (Simplicial sets) Every truncation $\Delta_n \to \Delta$ induces a level $\widehat{\Delta_n} \to \widehat{\Delta}$. (Every level of $\widehat{\Delta}$ is of this form.)

Some simple examples

Proposition

If $\mathcal{C}$ is a small category then restriction along a full inclusion $\mathcal{B} \to \mathcal{C}$ is the inverse image of a level $\widehat{\mathcal{B}} \to \widehat{\mathcal{C}}$.

1. (Simplicial sets) Every truncation $\Delta_n \to \Delta$ induces a level $\widehat{\Delta_n} \to \widehat{\Delta}$. (Every level of $\widehat{\Delta}$ is of this form.)
2. (The classifier of (non-trivial) Boolean algebras.) Let $\mathbb{F}$ be the category of nonempty finite sets and let $\mathbb{F}_n \to \mathbb{F}$ be the obvious truncation.

Some simple examples

Proposition

If $\mathcal{C}$ is a small category then restriction along a full inclusion $\mathcal{B} \to \mathcal{C}$ is the inverse image of a level $\widehat{\mathcal{B}} \to \widehat{\mathcal{C}}$.

1. (Simplicial sets) Every truncation $\Delta_n \to \Delta$ induces a level $\widehat{\Delta_n} \to \widehat{\Delta}$. (Every level of $\widehat{\Delta}$ is of this form.)
2. (The classifier of (non-trivial) Boolean algebras.) Let $\mathbb{F}$ be the category of nonempty finite sets and let $\mathbb{F}_n \to \mathbb{F}$ be the obvious truncation.
3. (Cubical? sets/The classifier of strictly bipointed objects) $[\mathcal{A}, \mathbf{Set}]$ where $\mathcal{A}$ is the category of finite strictly bipointed sets. Truncations.

Some simple examples

Proposition

If $\mathcal{C}$ is a small category then restriction along a full inclusion $\mathcal{B} \to \mathcal{C}$ is the inverse image of a level $\widehat{\mathcal{B}} \to \widehat{\mathcal{C}}$.

1. (Simplicial sets) Every truncation $\Delta_n \to \Delta$ induces a level $\widehat{\Delta_n} \to \widehat{\Delta}$. (Every level of $\widehat{\Delta}$ is of this form.)
2. (The classifier of (non-trivial) Boolean algebras.) Let $\mathbb{F}$ be the category of nonempty finite sets and let $\mathbb{F}_n \to \mathbb{F}$ be the obvious truncation.
3. (Cubical? sets/The classifier of strictly bipointed objects) $[\mathcal{A}, \mathbf{Set}]$ where $\mathcal{A}$ is the category of finite strictly bipointed sets. Truncations.
4. Other.

Remark:

Some simple examples

Proposition

If $\mathcal{C}$ is a small category then restriction along a full inclusion $\mathcal{B} \to \mathcal{C}$ is the inverse image of a level $\widehat{\mathcal{B}} \to \widehat{\mathcal{C}}$.

1. (Simplicial sets) Every truncation $\Delta_n \to \Delta$ induces a level $\widehat{\Delta_n} \to \widehat{\Delta}$. (Every level of $\widehat{\Delta}$ is of this form.)
2. (The classifier of (non-trivial) Boolean algebras.) Let $\mathbb{F}$ be the category of nonempty finite sets and let $\mathbb{F}_n \to \mathbb{F}$ be the obvious truncation.
3. (Cubical? sets/The classifier of strictly bipointed objects) $[\mathcal{A}, \mathbf{Set}]$ where $\mathcal{A}$ is the category of finite strictly bipointed sets. Truncations.
4. Other.

Remark: All the levels indicated above have monic skeleta.

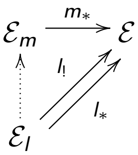
Aufhebung

The levels of $\mathcal{E}$ may be partially ordered as subtoposes.
That is, m is above l if and only if l_* factors through m_*
(I.e. l-sheaves are m-sheaves)

Aufhebung

The levels of $\mathcal{E}$ may be partially ordered as subtoposes.
That is, m is above l if and only if l_* factors through m_*
(I.e. l-sheaves are m-sheaves)

A level m is way above level l if both subcategories $l_!, l_* : \mathcal{E}_l \to \mathcal{E}$ factor through $m_* : \mathcal{E}_m \to \mathcal{E}$.

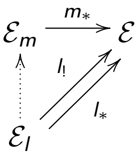


(I.e. both l-sheaves and l-skeletal objects are m-sheaves.)

Aufhebung

The levels of $\mathcal{E}$ may be partially ordered as subtoposes.
That is, m is above l if and only if l_* factors through m_*
(I.e. l-sheaves are m-sheaves)

A level m is way above level l if both subcategories $l_!, l_* : \mathcal{E}_l \to \mathcal{E}$ factor through $m_* : \mathcal{E}_m \to \mathcal{E}$.



(I.e. both l-sheaves and l-skeletal objects are m-sheaves.)

Definition

The Aufhebung of level l is the least level of $\mathcal{E}$ that is way above l.

Pre-cohesive geometric morphisms and level 0

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \big\downarrow \dashv \big\uparrow p^* \dashv \big\downarrow p_* \dashv \big\uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful,

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \downarrow \dashv \uparrow p^* \dashv \downarrow p_* \dashv \uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \Big\downarrow \dashv \Big\uparrow p^* \dashv \Big\downarrow p_* \dashv \Big\uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_*X) \to X$ is monic. Intuition:

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \Big\downarrow \dashv \Big\uparrow p^* \dashv \Big\downarrow p_* \dashv \Big\uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_* X) \to X$ is monic. Intuition:

pieces $\dashv$ discrete $\dashv$ points $\dashv$ codiscrete

Examples:

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \Big\downarrow \dashv \Big\uparrow p^* \dashv \Big\downarrow p_* \dashv \Big\uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_* X) \to X$ is monic. Intuition:

pieces $\dashv$ discrete $\dashv$ points $\dashv$ codiscrete

Examples: $\widehat{\Delta} \to \mathbf{Set}$,

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \downarrow \dashv \uparrow p^* \dashv \downarrow p_* \dashv \uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_* X) \to X$ is monic. Intuition:

pieces $\dashv$ discrete $\dashv$ points $\dashv$ codiscrete

Examples: $\widehat{\Delta} \to \mathbf{Set}$, its truncations $\widehat{\Delta_n} \to \mathbf{Set}$,

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \downarrow \dashv \uparrow p^* \dashv \downarrow p_* \dashv \uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_* X) \to X$ is monic. Intuition:

pieces $\dashv$ discrete $\dashv$ points $\dashv$ codiscrete

Examples: $\widehat{\Delta} \to \mathbf{Set}$, its truncations $\widehat{\Delta_n} \to \mathbf{Set}$, the classifier of BAs and its truncations,

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \Big\downarrow \dashv \Big\uparrow p^* \dashv \Big\downarrow p_* \dashv \Big\uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_* X) \to X$ is monic. Intuition:

$$\text{pieces} \quad \dashv \quad \text{discrete} \quad \dashv \quad \text{points} \quad \dashv \quad \text{codiscrete}$$

Examples: $\widehat{\Delta} \to \mathbf{Set}$, its truncations $\widehat{\Delta_n} \to \mathbf{Set}$, the classifier of BAs and its truncations, Cubical Sets,

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \Big\downarrow \;\dashv\; \Big\uparrow p^* \;\dashv\; \Big\downarrow p_* \;\dashv\; \Big\uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_* X) \to X$ is monic. Intuition:

$$\text{pieces} \quad \dashv \quad \text{discrete} \quad \dashv \quad \text{points} \quad \dashv \quad \text{codiscrete}$$

Examples: $\widehat{\Delta} \to \mathbf{Set}$, its truncations $\widehat{\Delta_n} \to \mathbf{Set}$, the classifier of BAs and its truncations, Cubical Sets, SDG,

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \Big\downarrow \;\dashv\; \Big\uparrow p^* \;\dashv\; \Big\downarrow p_* \;\dashv\; \Big\uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_* X) \to X$ is monic. Intuition:

pieces $\dashv$ discrete $\dashv$ points $\dashv$ codiscrete

Examples: $\widehat{\Delta} \to \mathbf{Set}$, its truncations $\widehat{\Delta_n} \to \mathbf{Set}$, the classifier of BAs and its truncations, Cubical Sets, SDG, AG, etc.

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \downarrow \dashv \uparrow p^* \dashv \downarrow p_* \dashv \uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_* X) \to X$ is monic. Intuition:

pieces $\dashv$ discrete $\dashv$ points $\dashv$ codiscrete

Examples: $\widehat{\Delta} \to$ **Set**, its truncations $\widehat{\Delta_n} \to$ **Set**, the classifier of BAs and its truncations, Cubical Sets, SDG, AG, etc.

The level $p^* \dashv p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ will be called Level 0.

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \downarrow \dashv \uparrow p^* \dashv \downarrow p_* \dashv \uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_* X) \to X$ is monic. Intuition:

pieces $\dashv$ discrete $\dashv$ points $\dashv$ codiscrete

Examples: $\widehat{\Delta} \to$ **Set**, its truncations $\widehat{\Delta_n} \to$ **Set**, the classifier of BAs and its truncations, Cubical Sets, SDG, AG, etc.

The level $p^* \dashv p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ will be called Level 0.
Notice that level 0 has monic skeleta (monic β).

Pre-cohesive geometric morphisms and level 0

Let $p : \mathcal{E} \to \mathcal{S}$ be a pre-cohesive geometric morphism, so that

$$\begin{array}{c} \mathcal{E} \\ p_! \downarrow \;\dashv\; \uparrow p^* \;\dashv\; \downarrow p_* \;\dashv\; \uparrow p^! \\ \mathcal{S} \end{array}$$

$p^*, p^! : \mathcal{S} \to \mathcal{E}$ are fully faithful, $p_!$ preserves finite products and the counit $\beta_X : p^*(p_* X) \to X$ is monic. Intuition:

pieces $\dashv$ discrete $\dashv$ points $\dashv$ codiscrete

Examples: $\widehat{\Delta} \to \mathbf{Set}$, its truncations $\widehat{\Delta_n} \to \mathbf{Set}$, the classifier of BAs and its truncations, Cubical Sets, SDG, AG, etc.

The level $p^* \dashv p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$ will be called Level 0.
Notice that level 0 has monic skeleta (monic β).
0-skeletal objects will be called discrete.

Level 1

Level 1

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Level 1

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Definition

Level 1 (of p) is the Aufhebung of level 0.

That is,

Level 1

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Definition

Level 1 (of p) is the Aufhebung of level 0.

That is, the least level of $\mathcal{E}$ that is way above level 0.

That is,

Level 1

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Definition

Level 1 (of p) is the Aufhebung of level 0.

That is, the least level of $\mathcal{E}$ that is way above level 0.

That is, the least level l of $\mathcal{E}$ such that discrete and codiscrete spaces are l-sheaves.

Lawvere's characterization of Level 1

"

Lawvere's characterization of Level 1

"Because of the special feature of dimension zero of having a components functor to it [...], the definition of dimension one is equivalent to the quite plausible condition:

Lawvere's characterization of Level 1

"Because of the special feature of dimension zero of having a components functor to it [...], the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of components of the skeleton at that dimension of the space" [L'91]

Lawvere's characterization of Level 1

"Because of the special feature of dimension zero of having a components functor to it [...], the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of components of the skeleton at that dimension of the space" [L'91]

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Theorem (Lawvere)

Lawvere's characterization of Level 1

"Because of the special feature of dimension zero of having a components functor to it [...], the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of components of the skeleton at that dimension of the space" [L'91]

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Theorem (Lawvere)

For any level l of $\mathcal{E}$ above level 0*,*

Lawvere's characterization of Level 1

"Because of the special feature of dimension zero of having a components functor to it [...], the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of components of the skeleton at that dimension of the space" [L'91]

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Theorem (Lawvere)

For any level l of $\mathcal{E}$ above level 0, l is way above 0 if and only if,

Lawvere's characterization of Level 1

"Because of the special feature of dimension zero of having a components functor to it [...], the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of components of the skeleton at that dimension of the space" [L'91]

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Theorem (Lawvere)

For any level l of $\mathcal{E}$ above level 0, l is way above 0 if and only if, for every X in $\mathcal{E}$,

Lawvere's characterization of Level 1

"Because of the special feature of dimension zero of having a components functor to it [...], the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of components of the skeleton at that dimension of the space" [L'91]

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Theorem (Lawvere)

For any level l of $\mathcal{E}$ above level 0, l is way above 0 if and only if, for every X in $\mathcal{E}$, $p_!(l_!(l^* X)) \to p_! X$ is an iso
(

Lawvere's characterization of Level 1

"Because of the special feature of dimension zero of having a components functor to it [...], the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of components of the skeleton at that dimension of the space" [L'91]

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Theorem (Lawvere)

For any level l of $\mathcal{E}$ above level 0, l is way above 0 if and only if, for every X in $\mathcal{E}$, $p_!(l_!(l^*X)) \to p_!X$ is an iso (where $l_!(l^*X) \to X$ is the l-skeleton of X).

"more pictorially:

Lawvere's characterization of Level 1

"Because of the special feature of dimension zero of having a components functor to it [...], the definition of dimension one is equivalent to the quite plausible condition: the smallest dimension such that the set of components of an arbitrary space is the same as the set of components of the skeleton at that dimension of the space" [L'91]

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with its associated level 0.

Theorem (Lawvere)

For any level l of $\mathcal{E}$ above level 0, l is way above 0 if and only if, for every X in $\mathcal{E}$, $p_!(l_!(l^*X)) \to p_!X$ is an iso (where $l_!(l^*X) \to X$ is the l-skeleton of X).

"more pictorially: if two points of any space can be connected by anything, then they can be connected by a curve."

Application: 1-dense subobjects with discrete domain

Application: 1-dense subobjects with discrete domain

Let A in $\mathcal{S}$ and $v : p^*A \to Y$ monic in $\mathcal{E}$.

Application: 1-dense subobjects with discrete domain

Let A in $\mathcal{S}$ and $v : p^*A \to Y$ monic in $\mathcal{E}$.

Lemma

If v is 1*-dense then*

Application: 1-dense subobjects with discrete domain

Let A in $\mathcal{S}$ and $v : p^*A \to Y$ monic in $\mathcal{E}$.

Lemma

If v is 1-dense then v is split.

Proof.

Application: 1-dense subobjects with discrete domain

Let A in $\mathcal{S}$ and $v : p^* A \to Y$ monic in $\mathcal{E}$.

Lemma

If v is 1-dense then v is split.

Proof.

By L's Thm. $p_! v : p_!(p^* A) \to p_! Y$ is an iso.

Application: 1-dense subobjects with discrete domain

Let A in $\mathcal{S}$ and $v : p^*A \to Y$ monic in $\mathcal{E}$.

Lemma

If v is 1-dense then v is split.

Proof.

By L's Thm. $p_! v : p_!(p^*A) \to p_! Y$ is an iso. Take the composite

$$Y \xrightarrow{\sigma} p^*(p_! Y) \xrightarrow{p^*(p_! v)^{-1}} p^*(p_!(p^*A)) \xrightarrow{p^*\tau} p^*A$$

where σ and τ are the unit and counit of $p_! \dashv p^*$. $\square$

Application: 1-dense subobjects with discrete domain

Let A in $\mathcal{S}$ and $v : p^*A \to Y$ monic in $\mathcal{E}$.

Lemma

If v is 1-dense then v is split.

Proof.

By L's Thm. $p_! v : p_!(p^*A) \to p_! Y$ is an iso. Take the composite

$$Y \xrightarrow{\sigma} p^*(p_! Y) \xrightarrow{p^*(p_! v)^{-1}} p^*(p_!(p^*A)) \xrightarrow{p^*\tau} p^*A$$

where σ and τ are the unit and counit of $p_! \dashv p^*$. $\square$

An object X in $\mathcal{E}$ is called (0-)separated if it is separated for the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$. (I.e. a subobject of a codiscrete object.)

Application: 1-dense subobjects with discrete domain

Let A in $\mathcal{S}$ and $v : p^*A \to Y$ monic in $\mathcal{E}$.

Lemma

If v is 1-dense then v is split.

Proof.

By L's Thm. $p_! v : p_!(p^*A) \to p_! Y$ is an iso. Take the composite

$$Y \xrightarrow{\sigma} p^*(p_! Y) \xrightarrow{p^*(p_! v)^{-1}} p^*(p_!(p^*A)) \xrightarrow{p^*\tau} p^*A$$

where σ and τ are the unit and counit of $p_! \dashv p^*$. $\square$

An object X in $\mathcal{E}$ is called (0-)separated if it is separated for the subtopos $p_* \dashv p^! : \mathcal{S} \to \mathcal{E}$. (I.e. a subobject of a codiscrete object.)

Lemma

If v is 1-dense and Y is separated then v is an iso.

Characterizations of skeletal objects

Characterizations of skeletal objects

Let m be the Aufhebung of level l.

Characterizations of skeletal objects

Let m be the Aufhebung of level l.
Can we characterize m-skeletal objects in terms of l?

Characterizations of skeletal objects

Let m be the Aufhebung of level l.
Can we characterize m-skeletal objects in terms of l?

Case $l = -\infty$ (so $m = 0$)

Characterizations of skeletal objects

Let m be the Aufhebung of level l.
Can we characterize m-skeletal objects in terms of l?

Case $l = -\infty$ (so $m = 0$)
(ct2018) If $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive, l.c. and $\mathcal{S}$ is Boolean then

Characterizations of skeletal objects

Let m be the Aufhebung of level l.
Can we characterize m-skeletal objects in terms of l?

Case $l = -\infty$ (so $m = 0$)
(ct2018) If $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive, l.c. and $\mathcal{S}$ is Boolean then X is 0-skeletal iff X is decidable.

Characterizations of skeletal objects

Let m be the Aufhebung of level l.
Can we characterize m-skeletal objects in terms of l?

Case $l = -\infty$ (so $m = 0$)
(ct2018) If $p : \mathcal{E} \to \mathcal{S}$ is pre-cohesive, l.c. and $\mathcal{S}$ is Boolean then X is 0-skeletal iff X is decidable.

Case $l = 0$ (so $m = 1$)
???

Bounded depth formulas

Bounded depth formulas

Consider the bounded depth formula

$$(\mathrm{BD}_1) \quad x_1 \vee (x_1 \Rightarrow (x_0 \vee \neg x_0))$$

(Bezhanishvili, Marra, McNeill, Pedrini, *Tarski's theorem on intuitionistic logic, for polyhedra*, APAL 2018)

Bounded depth formulas

Consider the bounded depth formula

$$(\mathrm{BD}_1) \quad x_1 \vee (x_1 \Rightarrow (x_0 \vee \neg x_0))$$

(Bezhanishvili, Marra, McNeill, Pedrini, *Tarski's theorem on intuitionistic logic, for polyhedra*, APAL 2018)

Notice that, assuming coHeyting operations one has

$$\begin{aligned} \top \leq x_1 \vee (x_1 \Rightarrow (x_0 \vee \neg x_0)) & \text{ iff } \top / x_1 \leq x_1 \Rightarrow (x_0 \vee \neg x_0) \\ & \text{ iff } (\top / x_1) \wedge x_1 \leq (x_0 \vee \neg x_0) \\ & \text{ iff } \partial x_1 \leq (x_0 \vee \neg x_0) \end{aligned}$$

Discrete boundaries

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with associated level 0.
Recall:

Discrete boundaries

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with associated level 0.
Recall: every X has monic 0-skeleton $\beta_X : p^*(p_*X) \to X$.
So,

Discrete boundaries

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with associated level 0.
Recall: every X has monic 0-skeleton $\beta_X : p^*(p_* X) \to X$.
So, for any subobject $u : U \to X$ we may build the implication
$u \Rightarrow \beta_X : (U \Rightarrow \beta_X) \to X$.

Discrete boundaries

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with associated level 0.
Recall: every X has monic 0-skeleton $\beta_X : p^*(p_*X) \to X$.
So, for any subobject $u : U \to X$ we may build the implication $u \Rightarrow \beta_X : (U \Rightarrow \beta_X) \to X$.

Definition

A subobject $u : U \to X$ has discrete boundary if $\top_X \leq u \vee (u \Rightarrow \beta_X)$ as subobjects of X. (Intuition:

Discrete boundaries

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with associated level 0.
Recall: every X has monic 0-skeleton $\beta_X : p^*(p_* X) \to X$.
So, for any subobject $u : U \to X$ we may build the implication $u \Rightarrow \beta_X : (U \Rightarrow \beta_X) \to X$.

Definition

A subobject $u : U \to X$ has discrete boundary if
$\top_X \leq u \vee (u \Rightarrow \beta_X)$ as subobjects of X. (Intuition: $\partial u \leq \beta_X$.)

Discrete boundaries

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with associated level 0.
Recall: every X has monic 0-skeleton $\beta_X : p^*(p_* X) \to X$.
So, for any subobject $u : U \to X$ we may build the implication
$u \Rightarrow \beta_X : (U \Rightarrow \beta_X) \to X$.

Definition

A subobject $u : U \to X$ has discrete boundary if
$\top_X \leq u \vee (u \Rightarrow \beta_X)$ as subobjects of X. (Intuition: $\partial u \leq \beta_X$.)

$$
\begin{array}{ccc}
p^* A = U \cap (U \Rightarrow \beta) & \longrightarrow & U \\
\downarrow & & \downarrow{\scriptstyle u} \\
(U \Rightarrow \beta) & \xrightarrow[u \Rightarrow \beta]{} & X
\end{array}
$$

Discrete boundaries

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with associated level 0.
Recall: every X has monic 0-skeleton $\beta_X : p^*(p_* X) \to X$.
So, for any subobject $u : U \to X$ we may build the implication $u \Rightarrow \beta_X : (U \Rightarrow \beta_X) \to X$.

Definition

A subobject $u : U \to X$ has discrete boundary if $\top_X \leq u \vee (u \Rightarrow \beta_X)$ as subobjects of X. (Intuition: $\partial u \leq \beta_X$.)

$$\begin{array}{ccc} p^* A = U \cap (U \Rightarrow \beta) & \longrightarrow & U \\ \downarrow & & \downarrow u \\ (U \Rightarrow \beta) & \xrightarrow[u \Rightarrow \beta]{} & X \end{array}$$

An object X in $\mathcal{E}$ has discrete boundaries if every subobject of X has discrete boundary.

The case of reflexive graphs

The case of reflexive graphs

Consider $p : \widehat{\Delta_1} \to \mathbf{Set}$.

The case of reflexive graphs

Consider $p : \widehat{\Delta_1} \to \mathbf{Set}$.

Proposition (somewhat misleading but suggestive statement)

The case of reflexive graphs

Consider $p : \widehat{\Delta_1} \to \mathbf{Set}$.

Proposition (somewhat misleading but suggestive statement)

A graph is 1-skeletal iff it has discrete boundaries.

The case of reflexive graphs

Consider $p : \widehat{\Delta_1} \to \mathbf{Set}$.

Proposition (somewhat misleading but suggestive statement)
A graph is 1-skeletal iff it has discrete boundaries.

Level 1 is the whole of $\widehat{\Delta_1}$.

The case of reflexive graphs

Consider $p : \widehat{\Delta_1} \to \mathbf{Set}$.

Proposition (somewhat misleading but suggestive statement)
A graph is 1-skeletal iff it has discrete boundaries.

Level 1 is the whole of $\widehat{\Delta_1}$.
Every graph has discrete boundaries.

The case of reflexive graphs

Consider $p : \widehat{\Delta_1} \to \mathbf{Set}$.

Proposition (somewhat misleading but suggestive statement)
A graph is 1-skeletal iff it has discrete boundaries.

Level 1 is the whole of $\widehat{\Delta_1}$.
Every graph has discrete boundaries.

Naive question: is it true in general that

1-skeletal iff discrete boundaries ?

The case of reflexive graphs

Consider $p : \widehat{\Delta_1} \to \mathbf{Set}$.

Proposition (somewhat misleading but suggestive statement)
A graph is 1-skeletal iff it has discrete boundaries.

Level 1 is the whole of $\widehat{\Delta_1}$.
Every graph has discrete boundaries.

Naive question: is it true in general that

1-skeletal iff discrete boundaries ?

No. See drawing.

The case of reflexive graphs

Consider $p : \widehat{\Delta_1} \to \mathbf{Set}$.

Proposition (somewhat misleading but suggestive statement)
A graph is 1-skeletal iff it has discrete boundaries.

Level 1 is the whole of $\widehat{\Delta_1}$.
Every graph has discrete boundaries.

Naive question: is it true in general that

1-skeletal iff discrete boundaries ?

No. See drawing.

Does anything survive the passage to the elementary setting?

Curves

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with associated level 0.

Curves

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with associated level 0.

Definition

An object X in $\mathcal{E}$ is a curve if there is an epic $Y \to X$ such that Y is separated and has discrete boundaries.

(Hence, curves have discrete boundaries.)

Curves

Let $p : \mathcal{E} \to \mathcal{S}$ be pre-cohesive with associated level 0.

Definition

An object X in $\mathcal{E}$ is a curve if there is an epic $Y \to X$ such that Y is separated and has discrete boundaries.

(Hence, curves have discrete boundaries.)

Consider Level 1 of $\mathcal{E}$.
(The least level s.t. codiscrete and discrete objects are sheaves.)

1-dense subobjects of sep. spaces with discrete boundaries

Lemma

Let $u : U \to Y$ be a 1-dense mono.

1-dense subobjects of sep. spaces with discrete boundaries

Lemma

Let $u : U \to Y$ be a 1-dense mono. If Y is separated and has discrete boundaries then u is an isomorphism.

Proof.

1-dense subobjects of sep. spaces with discrete boundaries

Lemma

Let $u : U \to Y$ be a 1-dense mono. If Y is separated and has discrete boundaries then u is an isomorphism.

Proof.

$$\begin{array}{ccc} p^* A & \longrightarrow & U \\ {\scriptstyle u'}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ U \Rightarrow \beta & \xrightarrow[u \Rightarrow \beta]{} & Y \end{array}$$

p.b. for some A in $\mathcal{S}$ so

1-dense subobjects of sep. spaces with discrete boundaries

Lemma

Let $u : U \to Y$ be a 1-dense mono. If Y is separated and has discrete boundaries then u is an isomorphism.

Proof.

$$\begin{array}{ccc} p^*A & \longrightarrow & U \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ U \Rightarrow \beta & \xrightarrow[u \Rightarrow \beta]{} & Y \end{array}$$

p.b. for some A in $\mathcal{S}$ so u' is an 1-dense subobject of $(U \Rightarrow \beta)$.

1-dense subobjects of sep. spaces with discrete boundaries

Lemma

Let $u : U \to Y$ be a 1-dense mono. If Y is separated and has discrete boundaries then u is an isomorphism.

Proof.

$$
\begin{array}{ccc}
p^*A & \longrightarrow & U \\
{\scriptstyle u'}\big\downarrow & & \big\downarrow{\scriptstyle u} \\
U \Rightarrow \beta & \xrightarrow[u \Rightarrow \beta]{} & Y
\end{array}
$$

p.b. for some A in $\mathcal{S}$ so u' is an 1-dense subobject of $(U \Rightarrow \beta)$.
$(U \Rightarrow \beta)$ is separated (it is a subobject of Y).
So

1-dense subobjects of sep. spaces with discrete boundaries

Lemma

Let $u : U \to Y$ be a 1-dense mono. If Y is separated and has discrete boundaries then u is an isomorphism.

Proof.

$$\begin{array}{ccc} p^*A & \longrightarrow & U \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ U \Rightarrow \beta & \underset{u \Rightarrow \beta}{\longrightarrow} & Y \end{array}$$

p.b. for some A in $\mathcal{S}$ so u' is an 1-dense subobject of $(U \Rightarrow \beta)$.
$(U \Rightarrow \beta)$ is separated (it is a subobject of Y).
So u' is an iso by the previous Lemma.

1-dense subobjects of sep. spaces with discrete boundaries

Lemma

Let $u : U \to Y$ be a 1-dense mono. If Y is separated and has discrete boundaries then u is an isomorphism.

Proof.

$$\begin{array}{ccc} p^*A & \longrightarrow & U \\ {\scriptstyle u'}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ U \Rightarrow \beta & \xrightarrow[u \Rightarrow \beta]{} & Y \end{array}$$

p.b. for some A in $\mathcal{S}$ so u' is an 1-dense subobject of $(U \Rightarrow \beta)$.
$(U \Rightarrow \beta)$ is separated (it is a subobject of Y).
So u' is an iso by the previous Lemma.
As Y has discrete boundaries (square is a p.o.) u is an iso. □

The elementary result: ‘curves are 1-dimensional’

The elementary result: 'curves are 1-dimensional'

Definition (Recall)

An object X in $\mathcal{E}$ is a curve if there is an epic $Y \to X$ such that Y is separated and has discrete boundaries.

The elementary result: 'curves are 1-dimensional'

Definition (Recall)

An object X in $\mathcal{E}$ is a curve if there is an epic $Y \to X$ such that Y is separated and has discrete boundaries.

Proposition

If level 1 of $\mathcal{E}$ has monic skeleta then curves 1-skeletal.

Proof.

The elementary result: ‘curves are 1-dimensional’

Definition (Recall)

An object X in $\mathcal{E}$ is a curve if there is an epic $Y \to X$ such that Y is separated and has discrete boundaries.

Proposition

If level 1 of $\mathcal{E}$ has monic skeleta then curves 1-skeletal.

Proof.

The previous Lemma implies that:

The elementary result: ‘curves are 1-dimensional’

Definition (Recall)

An object X in $\mathcal{E}$ is a curve if there is an epic $Y \to X$ such that Y is separated and has discrete boundaries.

Proposition

If level 1 of $\mathcal{E}$ has monic skeleta then curves 1-skeletal.

Proof.

The previous Lemma implies that: if Y is separated and has discrete boundaries then the 1-skeleton of Y is epic. □

The elementary result: ‘curves are 1-dimensional’

Definition (Recall)

An object X in $\mathcal{E}$ is a curve if there is an epic $Y \to X$ such that Y is separated and has discrete boundaries.

Proposition

If level 1 of $\mathcal{E}$ has monic skeleta then curves 1-skeletal.

Proof.

The previous Lemma implies that: if Y is separated and has discrete boundaries then the 1-skeleton of Y is epic. □

‘Proposition’

In the examples, an object is 1-skeletal if and only if it is a curve.

The case of presheaf toposes

Let $\mathcal{C}$ be a small category with terminal object and such that every object has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive.

Lemma

For any object C in $\mathcal{C}$, the following are equivalent:

1. *The representable $\mathcal{C}(\_, C)$ in $\widehat{\mathcal{C}}$ has discrete boundaries.*

The case of presheaf toposes

Let $\mathcal{C}$ be a small category with terminal object and such that every object has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive.

Lemma

For any object C in $\mathcal{C}$, the following are equivalent:

1. *The representable $\mathcal{C}(\_, C)$ in $\widehat{\mathcal{C}}$ has discrete boundaries.*
2. *For every $f : B \to C$ in $\mathcal{C}$, f is constant or f has a section.*

Such objects of $\mathcal{C}$ will be called edge types.

The case of presheaf toposes

Let $\mathcal{C}$ be a small category with terminal object and such that every object has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive.

Lemma

For any object C in $\mathcal{C}$, the following are equivalent:

1. *The representable $\mathcal{C}(\_, C)$ in $\widehat{\mathcal{C}}$ has discrete boundaries.*
2. *For every $f : B \to C$ in $\mathcal{C}$, f is constant or f has a section.*

Such objects of $\mathcal{C}$ will be called edge types.
Let $\mathcal{C}_{\mathrm{e}} \to \mathcal{C}$ be the full subcategory of edge-types.

Proposition

The case of presheaf toposes

Let $\mathcal{C}$ be a small category with terminal object and such that every object has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive.

Lemma

For any object C in $\mathcal{C}$, the following are equivalent:

1. *The representable $\mathcal{C}(\_, C)$ in $\widehat{\mathcal{C}}$ has discrete boundaries.*
2. *For every $f : B \to C$ in $\mathcal{C}$, f is constant or f has a section.*

Such objects of $\mathcal{C}$ will be called edge types.
Let $\mathcal{C}_{\mathrm{e}} \to \mathcal{C}$ be the full subcategory of edge-types.

Proposition

If points in $\mathcal{C}$ separate maps with edge type codomain and every object is edge-wise connected then $\widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ is level 1.

The case of presheaf toposes

Let $\mathcal{C}$ be a small category with terminal object and such that every object has a point so that $p : \widehat{\mathcal{C}} \to \mathbf{Set}$ is pre-cohesive.

Lemma

For any object C in $\mathcal{C}$, the following are equivalent:

1. *The representable $\mathcal{C}(\_, C)$ in $\widehat{\mathcal{C}}$ has discrete boundaries.*
2. *For every $f : B \to C$ in $\mathcal{C}$, f is constant or f has a section.*

Such objects of $\mathcal{C}$ will be called edge types.
Let $\mathcal{C}_{\mathrm{e}} \to \mathcal{C}$ be the full subcategory of edge-types.

Proposition

If points in $\mathcal{C}$ separate maps with edge type codomain and every object is edge-wise connected then $\widehat{\mathcal{C}_{\mathrm{e}}} \to \widehat{\mathcal{C}}$ is level 1. If, moreover, this level has monic skeleta then 1-skeletal objects coincide with curves.

Bibliography I

F. W. Lawvere.
Some thoughts on the future of category theory.
SLNM 1488, 1991.

N. Bezhanishvili, V. Marra, D. McNeill, and A. Pedrini.
Tarskis theorem on intuitionistic logic, for polyhedra.
Ann. Pure Appl. Logic, 2018.

F. W. Lawvere.
Linearization of graphic toposes via Coxeter groups.
JPAA, 2002.

G. M. Kelly and F. W. Lawvere.
On the complete lattice of essential localizations.
Bull. Soc. Math. Belg. Ser. A, 1989.

Bibliography II

C. Kennett, E. Riehl, M. Roy, and M. Zaks.
Levels in the toposes of simplicial sets and cubical sets.
J. Pure Appl. Algebra, 2011.

M. Menni
The Unity and Identity of decidable objects and double negation sheaves.
JSL, 2018.