Artin glueings of frames and toposes as semidirect products

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Artin glueings of topological spaces

Let $N = (|N|, \mathcal{O}(N))$ and $H = (|H|, \mathcal{O}(H))$ be topological spaces.

What topological spaces $G = (|G|, \mathcal{O}(G))$ satisfy that H is an open subspace and N its closed complement?

Such a space G we call an Artin glueing of H by N.

It must be that $|G| = |N| \sqcup |H|$.

Each open $U \in \mathcal{O}(G)$ then corresponds to a pair (U_N, U_H) where $U_N = U \cap N$ and $U_H = U \cap H$.

Thus $\mathcal{O}(G)$ is isomorphic to the frame L_G of certain pairs (U_N, U_H) with componentwise union and intersection.

The associated meet-preserving map

For each $U \in \mathcal{O}(H)$ there is a largest open $V \in \mathcal{O}(N)$ such that (V, U) occurs in L_G.

Let $f_G \colon \mathcal{O}(H) \to \mathcal{O}(N)$ be a function which assigns to each $U \in \mathcal{O}(H)$ the largest V.

This function preserves finite meets.

We have that $(V, U) \in L_G$ if and only if $V \subseteq f(U)$.

Given any finite-meet preserving map $f \colon \mathcal{O}(H) \to \mathcal{O}(N)$ we can construct a frame $\mathrm{Gl}(f)$ as above.

This frame $\mathrm{Gl}(f)$ will satisfy the required properties and we call it the Artin glueing of f.

Semidirect products of groups are analogous

Let N and H be groups.

A semidirect product of H by N is a group G satisfying the following conditions.

1. H is a subgroup and N a normal subgroup.
2. $N \cap H = \{e\}$.
3. $N \vee H = NH = \{nh : n \in N, h \in H\} = G$.

Conditions 2 and 3 together say that N and H are complements.

Every such group G is determined by a unique group homomorphism $\alpha\colon H \to \mathrm{Aut}(N)$.

Split extensions of groups

A diagram $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ is a split extension when the following hold:

1. k is the kernel of e,
2. e is the cokernel of k,
3. $se = \mathrm{id}$.

Every semidirect product of N and H yields a split extension.

Every split extension is of this form.

Every element $g \in G$ can be written $g = k(n)s(h)$ for $n \in N$ and $h \in H$. Thus G is generated by the images of k and s.

Artin glueings of frames as extensions

We want a pointed category in which there is a class of extensions of H by N which are precisely the Artin glueings of N and H.

The usual category of locales is no good as it does not have zero morphisms.

Instead consider the category RFrm whose objects are frames and whose morphisms are finite-meet preserving maps.

The maps $\top_{X,Y} \colon X \to Y$ sending each element of X to the top element 1 of Y form a class of zero morphisms.

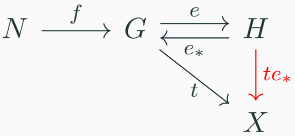
Cokernels

The cokernel of a morphism $f\colon N \to G$ in RFrm exists and is $e\colon G \to {\downarrow} f(0)$, where $e(g) = g \wedge f(0)$.

We call such a map a normal epimorphism.

Furthermore e has a right adjoint section $e_*(x) = (f(0) \Rightarrow x)$.

- Let $t\colon G \to X$ be such that $tf = \top$.



- If $e(x) = e(y)$ (i.e. $x \wedge f(0) = y \wedge f(0)$) then

  $$t(x) = t(x) \wedge 1 = t(x) \wedge t(f(0)) = t(x \wedge f(0)) = t(y \wedge f(0)) = t(y)$$

- Because $e(e_* e(x)) = e(x)$ we have that $t(e_* e(x)) = t(x)$.
- Thus $t = te_* e$ and te_* is the unique such map as e is epic.

Kernels

Kernels do not always exist in RFrm, but kernels of normal epis do.

The kernel of a normal epi $e\colon G \to {\downarrow}u$ is the inclusion $k\colon {\uparrow}u \hookrightarrow G$.

It has a left adjoint $k^*(x) = x \vee u$.

Split extensions in RFrm are of the form ${\uparrow}u \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} {\downarrow}u$.

Notice that ${\uparrow}u$ is the closed sublocale corresponding to u and that ${\downarrow}u$ is isomorphic to the open sublocale corresponding to u.

Only the splitting remains mysterious.

Schreier-type extensions and the splitting

Recall that if $N \overset{k}{\rightarrowtail} G \underset{s}{\overset{e}{\rightleftarrows}} H$ is a split extension of groups, then each element of G can be written $g = k(n)s(h)$.

For split extensions of monoids this does always hold. Split extensions where this holds are called weakly Schreier.

In RFrm a split extension will be weakly Schreier if and only if s is the right adjoint of e.

Thus a weakly Schreier extension is always of the form $\uparrow u \overset{k}{\rightarrowtail} G \underset{e_*}{\overset{e}{\rightleftarrows}} \downarrow u$. It is entirely determined by the normal epi e.

Artin glueings are weakly Schreier

Recall that the Artin glueing $\mathrm{Gl}(f)$ of a finite-meet preserving map $f\colon H \to N$ is the frame of pairs (n, h) where $n \le f(h)$.

We have projections $\pi_1\colon \mathrm{Gl}(f) \to N$ and $\pi_2\colon \mathrm{Gl}(f) \to H$ which preserve finite meets.

These projections have right adjoints which lie in RFrm.

1. $\pi_{1*}(n) = (n, 1)$.
2. $\pi_{2*}(h) = (f(h), h)$.

We have that $N \overset{\pi_{1*}}{\rightarrowtail} \mathrm{Gl}(f) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} H$ is a weakly Schreier extension.

Think of π_2 as the map $(-) \wedge (0,1)\colon \mathrm{Gl}(f) \to {\downarrow}(0,1)$

Notice that f can be recovered via $f = \pi_1 \pi_{2*}$.

All extensions are Artin glueings

From a weakly Schreier extension $N \overset{k}{\rightarrowtail} G \underset{e_*}{\overset{e}{\rightleftarrows}} H$ we can form

$$N \overset{\pi_{1*}}{\rightarrowtail} \mathrm{Gl}(k^* e_*) \underset{\pi_{2*}}{\overset{\pi_2}{\rightleftarrows}} H.$$

The frames G and $\mathrm{Gl}(k^* e_*)$ are isomorphic

1. $f \colon G \to \mathrm{Gl}(k^* e_*)$ sends g to $(k^*(g), e(g))$,
2. $f^{-1} \colon \mathrm{Gl}(k^* e_*) \to G$ sends (n, h) to $k(n) \wedge e_*(h)$.

Furthermore f and f^{-1} make the three squares below commute.

$$\begin{array}{ccccc}
N & \xrightarrow{\quad k \quad} & G & \underset{e_*}{\overset{e}{\rightleftarrows}} & H \\
\| & & {\scriptstyle f^{-1}}\uparrow\downarrow{\scriptstyle f} & & \| \\
N & \xrightarrow{\pi_{1*}} & \mathrm{Gl}(k^* e_*) & \underset{\pi_{1*}}{\overset{\pi_1}{\rightleftarrows}} & H
\end{array}$$

Further thoughts

Since Artin glueings correspond to finite-meet preserving maps, we have that $\mathrm{Hom}(-,-)$ is the bifunctor of weakly Schreier extensions.

The homsets of RFrm are actually meet-semilattices. This operation gives a Baer sum on our extensions.

P. Faul and G. Manuell. *Artin glueings of frames as semidirect products*: https://arxiv.org/abs/1907.05104

The Artin glueing construction works on toposes and there is an analogue on that level.

Artin glueings of toposes

Given toposes $\mathbf{N}$ and $\mathbf{H}$ we can ask for which toposes $\mathbf{G}$ is $\mathbf{H}$ an open subtopos and $\mathbf{N}$ its closed complement.

Such toposes will always be determined by a left exact functor $F\colon \mathbf{H} \to \mathbf{N}$ in the following way.

- The objects are triples (N, H, ℓ) such that $\ell\colon N \to F(H)$
- The morphisms are pairs $(f, g)\colon (N, H, \ell) \to (N', H', \ell')$ such that

$$\begin{array}{ccc} N & \xrightarrow{\quad f \quad} & N' \\ \big\downarrow{\scriptstyle \ell} & & \big\downarrow{\scriptstyle \ell'} \\ F(H) & \xrightarrow{F(g)} & F(H') \end{array}$$

  commutes.

We write $\mathbf{Gl}(F)$ and call this topos the Artin glueing along F.

Categorifying the idea for frames

We can consider the 2-category RTopos of toposes with left exact functors and natural transformations.

Between any two toposes there are left exact functors that send every object in the domain to a terminal object in the codomain. These behave like zero morphisms.

We can define the 2-cokernel of a left exact functor F as the 2-coequaliser of F with a zero morphism. Similarly we can define the 2-kernel.

2-Cokernels and 2-kernels

2-Cokernels always exist.

Given a left exact functor F the 2-cokernel is the open subtopos $\mathfrak{O}_{F(0)}$ corresponding to the subterminal $F(0)$.

2-Kernels in general do not exist but 2-kernels of 2-cokernels $E\colon \mathbf{G} \to \mathfrak{O}_U$ do exist.

It is given by the inclusion of the full subcategory of objects sent to a terminal object by E.

This turns out to be the inclusion $K\colon \mathfrak{C}_U \to \mathbf{G}$ of the closed subtopos corresponding to the subterminal U.

Adjoint extensions

An adjoint extension is a diagram $\mathbf{N} \overset{K}{\rightarrowtriangle} \mathbf{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathbf{H}$ where

1. K is the 2-kernel of E and
2. E is the 2-cokernel of K.

We see that the adjoint extensions in RTopos are all of the form $\mathfrak{C}_U \overset{K}{\rightarrowtriangle} \mathbf{G} \underset{E_*}{\overset{E}{\rightleftarrows}} \mathfrak{O}_U$.

But this is the inclusion of complemented subtoposes, so this is an Artin glueings.

$\mathbf{G}$ is given by $\mathrm{Gl}(K^* E_*)$.

Further thoughts

Similarly we can think of Hom as giving an extension bifunctor.

Since $\mathrm{Hom}(-,-)$ returns finitely complete categories we should be able to compute limits of extensions.

There seems to be a relationship to recollements, the additive (or triangulated) version of Artin glueings.