Internal lenses as monad morphisms

Bryce Clarke
CoACT, Macquarie University, bryce.clarke1@students.mq.edu.au

CT2019, July 12

Outline

Motivation

Formal monads, functors, and cofunctors

The 2-category of internal lenses

Bonus: Mealy morphisms and symmetric lenses

Motivation

What is a lens?

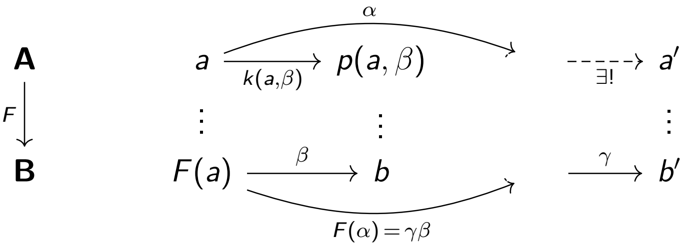
- It depends on who you ask . . .
- Lenses are a mathematical structure which aim to capture the notion of synchronisation between a pair of systems (e.g. databases).
- Originally lenses were defined to precisely present synchronisation between a pair of sets, called state-based lenses.
- However when generalising to categories, there are at least two useful ways to define lenses: split opfibrations[1] and (asymmetric) delta lenses[2].
- In this talk, I will show how delta lenses may be internalised to a category with pullbacks and motivated as a kind of monad morphism.
- Link: Internal lenses as functors and cofunctors, *Proceedings of ACT2019*.

[1] Called "c-lenses" in: Johnson, Rosebrugh, Wood, *Lenses, fibrations and universal translations* (2012)

[2] Diskin, Xiong, & Czarnecki, *From State- to Delta-Based Bidirectional Model Transformations: the Asymmetric Case* (2011)

Split opfibrations

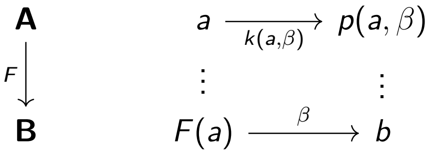
MACQUARIE University



A split opfibration is a functor $F\colon \mathbf{A} \to \mathbf{B}$ such that for all pairs (a, β) as shown, there is a morphism $k(a,\beta)\colon a \to p(a,\beta)$ with $F(p(a,\beta)) = b$ such that:

1. $Fk(a,\beta) = \beta$
2. The morphisms $k(a,\beta)$ are opcartesian, satisfying the universal property illustrated above, thus defining a cleavage.
3. The cleavage respects identities and composition, thus defining a splitting.

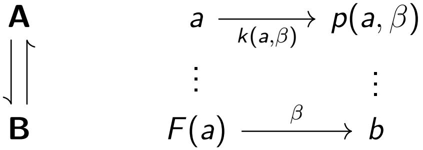
Delta lenses



A delta lens is a functor $F \colon \mathbf{A} \to \mathbf{B}$ such that for all pairs (a, β) as shown, there is a morphism $k(a, \beta) \colon a \to p(a, \beta)$ with $F(p(a, \beta)) = b$ such that:

1. $Fk(a, \beta) = \beta$
2.
3. The lifting k respects identities and composition.

Cofunctors (not contravariant functors!)



A cofunctor is an *function* $F\colon A_0 \to B_0$ such that for all pairs (a, β) as shown, there is a morphism $k(a, \beta)\colon a \to p(a, \beta)$ with $F(p(a, \beta)) = b$ such that:

1.
2.

3. The lifting k respects identities and composition.

A brief history of cofunctors

- First defined by Higgins & Mackenzie[3] as "comorphisms" between vector bundles, modules, Lie algebroids, and Lie pseudo-algebras.
- Generalised by Aguiar[4] where "cofunctors" are between categories internal to a monoidal category with equalizers.
- Rediscovered by Ahman & Uustalu[5] as directed container morphisms or "split pre-opcleavages".
- Considered recently in talks by Garner (groupoids and cofunctors)[6], Cockett (internal partite categories and cofunctors)[7] and Paré (Retrocells, CT2019).

[3] *Duality for base-changing morphisms of vector bundles, modules, Lie algebroids and Poisson structures* (1993)

[4] *Internal Categories and Quantum Groups* (1997)

[5] *Directed Containers as Categories* (2016)

[6] *Inner automorphisms of groupoids*, Australian Category Seminar (13 March 2019)

[7] *Hyperconnections*, Australian Category Seminar (20 March 2019)

Formal monads, functors, and cofunctors

A review of formal monads

A monad[8] (A, t) in a 2-category $\mathcal{K}$ consists of an object A, a 1-cell $t\colon A \to A$ and 2-cells $\eta\colon 1_A \to t$ and $\mu\colon t^2 \to t$ such that the following diagrams commute:

$$
\begin{array}{ccccc}
t & \xrightarrow{t\eta} & t^2 & \xleftarrow{\eta t} & t \\
& {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1} & \\
& & t & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
t^3 & \xrightarrow{t\mu} & t^2 \\
{\scriptstyle \mu t}\downarrow & & \downarrow{\scriptstyle \mu} \\
t^2 & \xrightarrow[\mu]{} & t
\end{array}
$$

We may similarly define a monad in a bicategory by inserting the unitors and associator where appropriate.

[8] Street, *The Formal Theory of Monads* (1972) and Lack, Street, *The Formal Theory of Monads II* (2002)

Let $\mathcal{E}$ be a category with pullbacks, and Span($\mathcal{E}$) the bicategory of spans in $\mathcal{E}$.

A monad in Span($\mathcal{E}$) consists of an object A_0 and a span,

$$\begin{array}{ccccc} & & A_1 & & \\ & \swarrow^{s_0} & & \searrow^{t_0} & \\ A_0 & & & & A_0 \end{array}$$

together with morphisms of spans,

$$\begin{array}{ccccc} & & A_0 & & \\ & \swarrow^{1_{A_0}} & \downarrow i & \searrow^{1_{A_0}} & \\ A_0 & \xleftarrow[s_0]{} & A_1 & \xrightarrow[t_0]{} & A_0 \end{array} \qquad \begin{array}{ccccc} A_1 & \xleftarrow{s_1} & A_2 & \xrightarrow{t_1} & A_1 \\ {\scriptstyle s_0}\downarrow & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle t_0} \\ A_0 & \xleftarrow[s_0]{} & A_1 & \xrightarrow[t_0]{} & A_0 \end{array}$$

satisfying some conditions. This defines an internal category in $\mathcal{E}$.

Lax and colax monad morphisms

A lax monad morphism $(A, t) \to (B, s)$ consists of a 1-cell $f\colon A \to B$ and a 2-cell $\phi\colon sf \to ft$ such that the following diagrams commute:

$$\begin{array}{ccc} & f & \\ {}^{\eta f}\swarrow & & \searrow^{f\eta} \\ sf & \xrightarrow[\phi]{} & ft \end{array} \qquad \begin{array}{ccccc} ssf & \xrightarrow{s\phi} & sft & \xrightarrow{\phi t} & ftt \\ {\scriptstyle \mu f}\downarrow & & & & \downarrow{\scriptstyle f\mu} \\ sf & & \xrightarrow[\phi]{} & & ft \end{array}$$

A colax monad morphism $(A, t) \to (B, s)$ consists of a 1-cell $f\colon A \to B$ and a 2-cell $\psi\colon ft \to sf$ such that the following diagrams commute:

$$\begin{array}{ccc} & f & \\ {}^{f\eta}\swarrow & & \searrow^{\eta f} \\ ft & \xrightarrow[\psi]{} & sf \end{array} \qquad \begin{array}{ccccc} ftt & \xrightarrow{\psi t} & sft & \xrightarrow{s\psi} & ssf \\ {\scriptstyle f\mu}\downarrow & & & & \downarrow{\scriptstyle \mu f} \\ ft & & \xrightarrow[\psi]{} & & sf \end{array}$$

Internal functors as monad morphisms in Span($\mathcal{E}$)

MACQUARIE University

- Since monads in Span($\mathcal{E}$) are internal categories, we might expect that the monad morphisms give internal functors ...but they do not (in general).
- Internal functors are colax monad morphisms whose 1-cell is a *left adjoint*.
- A left adjoint 1-cell in Span($\mathcal{E}$) is a span whose left leg is an identity:

$$\begin{array}{ccccc} & & A_0 & & \\ & {\scriptstyle 1_{A_0}}\swarrow & & \searrow{\scriptstyle f_0} & \\ A_0 & & & & B_0 \end{array}$$

- The corresponding 2-cell simplifies to a morphism $f_1 \colon A_1 \to B_1$ satisfying:

$$\begin{array}{ccccc} A_0 & \xleftarrow{s_0} & A_1 & \xrightarrow{t_0} & A_0 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ B_0 & \xleftarrow[s_0]{} & B_1 & \xrightarrow[t_0]{} & B_0 \end{array}$$

What are the lax monad morphisms in $\mathsf{Span}(\mathcal{E})$?

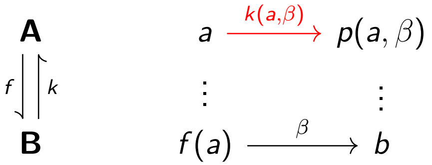
- Given that internal functors are important, we may also wish to consider the corresponding dual notion.
- Internal cofunctors are lax monad morphisms whose 1-cell is a left adjoint.
- The corresponding 2-cell for an internal cofunctor amounts to a morphism $k_1 \colon A_0 \times_{B_0} B_1 \to A_1$ making the diagram commmute:

$$
\begin{array}{ccccc}
A_0 & \xleftarrow{\;s_0\;} & A_1 & \xrightarrow{\;t_0\;} & A_0 \\
\uparrow{\scriptstyle 1_{A_0}} & & \uparrow{\scriptstyle k_1} & & \uparrow{\scriptstyle 1_{A_0}} \\
A_0 & \xleftarrow{\;\pi_0\;} & A_0 \times_{B_0} B_1 & \xrightarrow{\;t_0 k_1\;} & A_0 \\
\downarrow{\scriptstyle f_0} & \lrcorner & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle f_0} \\
B_0 & \xleftarrow[\;s_0\;]{} & B_1 & \xrightarrow[\;t_0\;]{} & B_0
\end{array}
$$

What is a cofunctor in Span(Set)?

MACQUARIE University

- A cofunctor $\Lambda \colon \mathbf{B} \rightleftharpoons \mathbf{A}$ should be understood as a kind of lifting.



- The codomain $p(a, \beta)$ of the lift should satisfy $f(p(a, \beta)) = b$.
- The lifting should respect both identities and composition.
- Examples of cofunctors include:
  - Discrete opfibrations and split opfibrations;
  - Identity-on-objects functors, such as monoid and group homomorphisms;
  - State-based lenses and delta lenses (!)

Cofunctors as spans of functors

- Every internal cofunctor may be represented as a span of internal functors,

$$\mathbf{B} \xleftarrow{\;\overline{F}\;} \mathbf{\Lambda} \xrightarrow{\;K\;} \mathbf{A}$$

where $\overline{F}$ is a discrete opfibration[9] and K is an identity-on-objects functor.

- These functors appear immediately in the definition of an internal cofunctor:

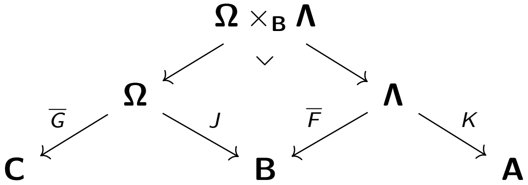
$$\begin{array}{ccccc}
A_0 & \xleftarrow{\;\pi_0\;} & A_0 \times_{B_0} B_1 & \xrightarrow{\;t_0 k_1\;} & A_0 \\
{\scriptstyle f_0}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle f_0} \\
B_0 & \xleftarrow[\;s_0\;]{} & B_1 & \xrightarrow[\;t_0\;]{} & B_0
\end{array}
\qquad
\begin{array}{ccccc}
A_0 & \xleftarrow{\;\pi_0\;} & A_0 \times_{B_0} B_1 & \xrightarrow{\;t_0 k_1\;} & A_0 \\
{\scriptstyle 1_{A_0}}\downarrow & & \downarrow{\scriptstyle k_1} & & \downarrow{\scriptstyle 1_{A_0}} \\
A_0 & \xleftarrow[\;s_0\;]{} & A_1 & \xrightarrow[\;t_0\;]{} & A_0
\end{array}$$

[9] Also called an *internal diagram*, an *internal $\mathcal{E}$-valued functor*, or an *internal copresheaf*.

Composition of cofunctors

MACQUARIE University

- Composing cofunctors is more difficult than composing functors; despite both arising from composition of the corresponding monad morphisms.
- However when a cofunctor is represented as a span of functors, composition is just via pullback.



- Every cofunctor may be factorised into a discrete opfibration part and an identity-on-objects functor part.

The 2-category of internal lenses

Internal lenses as monad morphisms

- An internal lens is a lax monad morphism $(A, t) \to (B, t)$ whose 1-cell $f\colon A \to B$ is a left adjoint and whose 2-cell $\phi\colon sf \rightleftarrows ft$ is a *section*.
- An internal lens is a functor (f_0, f_1) and a cofunctor (f_0, k_1) satisfying:

$$
\begin{array}{ccc}
 & A_0 \times_{B_0} B_1 & \\
{}^{k_1}\swarrow & & \searrow^{\pi_1} \\
A_1 & \xrightarrow[f_1]{} & B_1
\end{array}
$$

- An internal lens is a commuting diagram of internal functors,

$$
\begin{array}{ccc}
 & \mathbf{\Lambda} & \\
{}^{K}\swarrow & & \searrow^{\overline{F}} \\
\mathbf{A} & \xrightarrow[F]{} & \mathbf{B}
\end{array}
$$

where $\overline{F}$ is a discrete opfibration and K is an identity-on-objects functor.

Examples of internal lenses

- A delta lens is exactly an internal lens in **Set**.
- A state-based lens consisting of functions,

$$g \colon S \to V \qquad\qquad p \colon S \times V \to S$$

  is a delta lens between codiscrete categories, where:

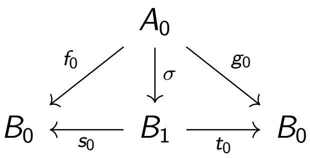
$$k_1 = \langle \pi_0, p \rangle \colon S \times V \to S \times S$$

- A delta lens between monoids is exactly a retraction.
- A discrete opfibration is an internal lens in **Set** where k_1 is an isomorphism.
- A split opfibration is an internal lens in **Cat** between double categories of squares, where k_1 is a left-adjoint right-inverse functor between categories:

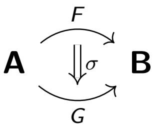
$$1 \circlearrowright (F \downarrow \mathbf{B}) \underset{\longleftarrow}{\overset{k_1}{\underset{\bot}{\longrightarrow}}} \mathbf{A}^2$$

Natural transformations between (co)functors

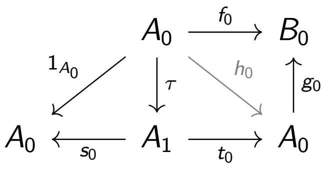
A natural transformation between internal functors $F, G \colon \mathbf{A} \to \mathbf{B}$ is given by a morphism $\sigma \colon A_0 \to B_1$ such that:



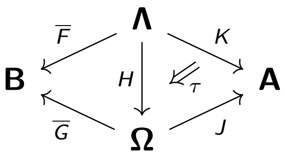
Represented by the diagram:



A natural transformation between internal cofunctors $\Lambda, \Omega \colon \mathbf{B} \rightleftharpoons \mathbf{A}$ is given by a morphism $\tau \colon A_0 \to A_1$ such that:

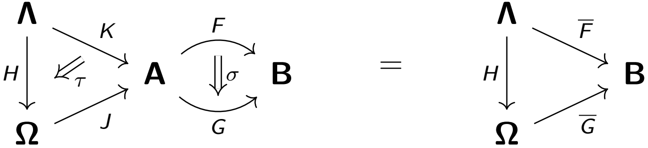


Actually just given by a diagram:



Natural transformations between lenses

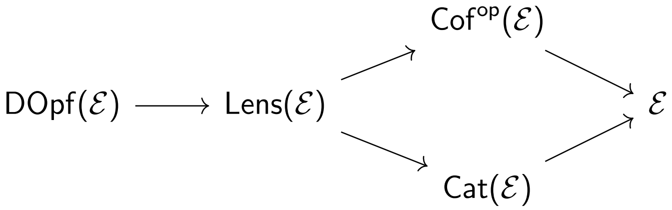
A natural transformation between internal lenses $(F, \Lambda), (G, \Omega) \colon \mathbf{A} \rightleftharpoons \mathbf{B}$ consists of natural transformations between the functor and cofunctor parts such that:



- We have a 2-category $\mathsf{Lens}(\mathcal{E})$ of whose objects are internal categories, morphisms are internal lenses, and 2-cells are natural transformations.
- There are forgetful 2-functors to the 2-categories $\mathsf{Cat}(\mathcal{E})$ and $\mathsf{Cof}(\mathcal{E})$.

Summary

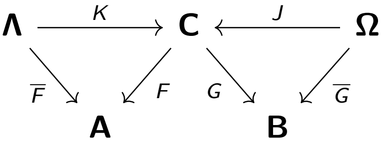
- Both cofunctors and lenses capture the notion of lifting morphisms between categories.
- Internal lenses are lax monad morphisms in $\mathsf{Span}(\mathcal{E})$ whose 1-cell is a left adjoint and whose 2-cell is a section.
- Every internal lens is a functor and a cofunctor, and may be represented as a particular commuting triangle of functors.
- There is a diagram of forgetful (2-)functors between (2-)categories:



Bonus: Mealy morphisms and symmetric lenses

Spans of internal lenses

- Internal lenses capture inherently asymmetric relationship, however often applications require a kind of symmetric synchronisation[10].
- The obvious way generalise is to consider morphisms from the bicategory Span(Lens($\mathcal{E}$)) whose 1-cells are spans of internal lenses . . .



- . . . however it is not known how to compute pullbacks in Lens($\mathcal{E}$).
- Fortunately we can canonically construct "fake pullbacks" of internal lenses which are sent to genuine pullbacks by the functor Lens($\mathcal{E}$) $\to$ Cat($\mathcal{E}$).

[10] Diskin, et al., *From State- to Delta-Based Bidirectional Model Transformations: The Symmetric Case* (2011)

- Given a span of internal lenses, we may compose the functors to obtain:

- We know how to compose in $\mathsf{Span}(\mathsf{Cat}(\mathcal{E}))$ and discrete opfibrations are stable under pullback – this could be a better notion for "symmetric lenses".
- Notice the above diagram is simply a pair of spans of functors whose:
  - left leg is a discrete opfibration;
  - underlying span of object assignments are opposite.
- These spans are similar to both functors and cofunctors, but what are they?

Mealy morphisms as lax monad morphisms

MACQUARIE University

- An (internal) Mealy morphism[11] is a lax monad morphism in Span($\mathcal{E}$).
- The 1-cell is a span in $\mathcal{E}$ while the 2-cell corresponds to the diagram:

$$\begin{array}{ccccc}
A_0 & \xleftarrow{s_0} & A_1 & \xrightarrow{t_0} & A_0 \\
\uparrow{\scriptstyle g_0} & & \uparrow{\scriptstyle g_1} & & \uparrow{\scriptstyle g_0} \\
X_0 & \xleftarrow{\pi_0} & X_0 \times_{B_0} B_1 & \xrightarrow{p_0} & X_0 \\
\downarrow{\scriptstyle f_0} & \lrcorner & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle f_0} \\
B_0 & \xleftarrow[s_0]{} & B_1 & \xrightarrow[t_0]{} & B_0
\end{array}$$

- Mealy morphisms are also known as two-dimensional partial maps between categories, and are classified by the *Fam construction*.

[11]Paré, *Mealy Morphisms of Enriched Categories* (2012)

Mealy morphisms as spans of functors

- Every Mealy morphism $\mathbf{B} \nrightarrow \mathbf{A}$ may be represented as a span of internal functors,

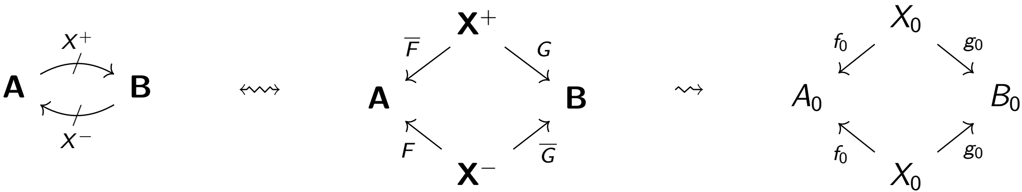
$$
\begin{array}{ccccc}
 & & \mathbf{X} & & \\
 & \overline{F} \swarrow & & \searrow G & \\
\mathbf{B} & & & & \mathbf{A}
\end{array}
$$

  where $\overline{F}$ is a discrete opfibration.
- Functors and cofunctors are both examples of Mealy morphisms.
- Using the (bo, ff)-factorisation in $\mathbf{Cat}(\mathcal{E})$, every Mealy morphism factorises into a cofunctor and a (fully faithful) functor, where $\mathrm{im}(F)$ is the *full image*.

$$
\begin{array}{ccccccccc}
 & & \mathbf{X} & & & & \mathrm{im}(G) & & \\
 & \overline{F} \swarrow & & \searrow G_{bo} & & 1 \swarrow & & \searrow G_{ff} & \\
\mathbf{B} & & & & \mathrm{im}(G) & & & & \mathbf{A}
\end{array}
$$

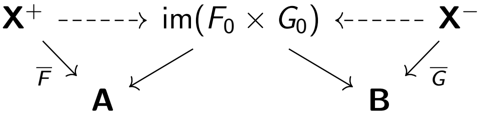
Symmetric lenses as Mealy morphisms

- A symmetric lens[12] is a pair of *compatible* Mealy morphisms:



Compatibility means the underlying object spans are opposite, as depicted.

- Every symmetric lens induces a span of internal lens,



via the (bo, ff)-factorisation of the functor $F_0 \times G_0 \colon \mathbf{X}_0 \to \mathbf{A} \times \mathbf{B}$.

[12] Johnson, Rosebrugh, *Symmetric Delta Lenses and Spans of Asymmetric Delta Lenses* (2017)

Summary (again)

MACQUARIE University

- Both cofunctors and lenses capture the notion of lifting morphisms between categories.
- Internal lenses are lax monad morphisms in $\mathsf{Span}(\mathcal{E})$ whose 1-cell is a left adjoint and whose 2-cell is a section.
- Every internal lens is a functor and a cofunctor, and may be represented as a particular commuting triangle of functors.
- Every symmetric lens is a compatible pair of Mealy morphisms, or equivalently, a span of internal lenses.
- More details can be found in the extended abstract linked below: Internal lenses as functors and cofunctors, *Proceedings of ACT2019*.