Categorical semantics of metric spaces and continuous logic

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Outline

- Motivation & perspective
- Brief review of continuous logic
- An examination of the category of metric spaces
  - Reverse engineering a "continuous semantics"
- A continuous subobject classifier in the style of Barr
- Categories of presheaves of metric spaces

Motivation & perspective

Lawvere noticed that metric spaces are categories enriched over

$$\mathbb{R}_{\geq 0} = (0 \leftarrow \cdots \leftarrow r \leftarrow \cdots \leftarrow \infty)$$

with tensor given by addition.

- $X(a,b)$ is "the distance from a to b", and the condition

  $$X(a,b) + X(b,c) \quad \longrightarrow \quad X(a,c)$$

  is the triangle inequality.

- $X(a,b)$ expresses degree of truth of the equality predicate on a and b
  - 0 is "true" and ∞ is "false"
  - The triangle inequality is the transitivity of equality

We work with $\mathbb{I} = (0 \leftarrow \cdots \leftarrow r \leftarrow \cdots \leftarrow 1)$ with truncated addition.

Motivation & perspective

On the one hand, *continuous logic* is a relatively new $[0,1]$-valued first order logic important to the model theory community.

- One reason for this is that surprisingly many of the (non-continuous) model theoretic notions have sensible continuous analogues.

On the other hand, categorical semantics has been extremely successful at analyzing the logical structure of categories.

Brief review of continuous logic

Continuous logic is the same as the usual first order logic, except:

- Sorts are interpreted as metric spaces (as opposed to as sets) with diameter ≤ 1.
- Function & predicate symbols come with a specified modulus of uniform continuity; their interpretations must obey the modulus
- Predicates are interpreted as uniformly continuous maps $X \to [0, 1]$ (as opposed to as set functions $X \to \{0, 1\}$)
- The distance function on a space X plays the role of the equality predicate
- Universal/existential quantification is sup/inf

So the interpretation of the syntax of continuous logic takes place in the category **Met** whose objects are metric spaces of diameter ≤ 1, and whose morphisms are uniformly continuous maps.

We work in the category **pMet**, which allows *pseudo*metric spaces.

The category of metric spaces

Categorical semantics informs us what structures are required of a given category in order to support various fragments of logic.

To get all of first order logic, sufficient to require the category be *geometric*

- has finite limits
- has images which are stable under pullback
- for each object X, $\mathrm{Sub}_m X$ is small-complete lattice with structure preserved by pullback
- interpret predicates on X as subobjects of X

Example: **Set**

The category of metric spaces

There is a variant using regular monos instead of monos: we require that the category

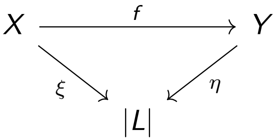
- has finite limits
- has regular images which are stable under pullback
- composing two regular monos yields a regular mono
- for each X, the lattice $\operatorname{Sub} X$ of regular subobjects is small-complete, with structure preserved by pullback
- interpret predicates on X as regular subobjects of X

Example: **pMet**

Barr established the following equivalence of categories, given a locale L:

$$\mathrm{Fuz}(L) \simeq \mathrm{Mon}(L^+)$$

- $\mathrm{Fuz}(L)$ is the category whose objects are set functions $\xi : X \to |L|$ and morphisms are (noncommutative) triangles



  for which $f \circ \eta \leq_{L^{\mathrm{op}}} \xi$
- $\mathrm{Mon}(L^+)$ is the category of sheaves of monos* on L^+, where $L^+ = L \cup \{i\}$ and the topology is the logic topology

Barr's equivalence

$$\mathrm{Fuz}(L) \simeq \mathrm{Mon}(L^+)$$

The actual maps are given by

$$\begin{array}{ccc} \xi & \mapsto & (r \mapsto \{x \mid \xi(x) \leq_{L^{op}} r\}) \\ (x \mapsto \inf\{r \in L^{op} \mid x \in R(r)\}) & \leftarrow\!\shortmid & R \end{array}$$

$R \in \mathrm{Mon}(L^+)$ is (up to iso) a meet-preserving functor $\underline{R} : L^{op} \to \mathrm{Sub}(R_{tot})$ where $R_{tot} = \bigcup_{r \in L^{op}} R(r)$.

The slogan is:

A function on X valued in $|L|$ is equivalent to the data of meet-preserving $(L^+)^{op}$-indexed sublevelsets of X.

Indexed subobjects

As a trivial special case of Barr's equivalence.

Write $\mathbb{2} = (0 \leftarrow 1)$ and $|\mathbb{2}| = \{0, 1\}$.

In **Set**, a predicate R on X is:

- Function $\chi_R : X \longrightarrow |\mathbb{2}|$
- Functor $R : \mathbb{2}^{\mathsf{op}} \longrightarrow \mathrm{Sub}\, X$ where

$$R(0) = \{x \in X \mid \chi_R(x) \leq_{\mathbb{2}^{\mathsf{op}}} 0\}$$
$$R(1) = \{x \in X \mid \chi_R(x) \leq_{\mathbb{2}^{\mathsf{op}}} 1\} = X,$$

i.e. a meet-preserving functor $R : \mathbb{2}^{\mathsf{op}} \to \mathrm{Sub}\, X$

So the subobject classifier in **Set** is just a Barr-style equivalence between functions into classical truth values and functors of subobjects on classical truth values.

Indexed subobjects

Recall $\mathbb{I} = 0 \leftarrow \cdots \leftarrow r \leftarrow \cdots \leftarrow 1$, and write $|\mathbb{I}| = [0, 1]$.

In continuous logic, predicates on $X \in$ **pMet** are uniformly continuous maps $X \to |\mathbb{I}|$. These should correspond to appropriate functors $\mathbb{I}^{\mathrm{op}} \to \operatorname{Sub} X$.

Given $f : X \to |\mathbb{I}|$, should look at functor $R_f : \mathbb{I}^{\mathrm{op}} \to \operatorname{Sub} X$ defined by

$$R_f(r) = \{x \in X \mid f(x) \leq_{\mathbb{I}^{\mathrm{op}}} r\}$$

Continuity of f should translate into some property of R_f...

Formalizing the metric

Given $X \in \mathbf{pMet}$ with metric d_X, have $D_X : \mathbb{I}^{\mathrm{op}} \to \mathrm{Sub}(X \times X)$ defined by

$$D_X(r) = \{(x, y) \in X \times X \mid d_X(x, y) \leq_{\mathbb{I}^{\mathrm{op}}} r\}$$

Proposition

There is a choice of distinguished $D_X : \mathbb{I}^{\mathrm{op}} \to \mathrm{Sub}(X \times X)$ for each $X \in \mathbf{pMet}$, as well as a choice of product $X \times Y$ for each $X, Y \in \mathbf{pMet}$, such that the following hold:

Formalizing the metric

- $D_X(0)$ contains the diagonal;
- The symmetry iso $X \times X \xrightarrow{\cong} X \times X$ takes D_X to itself;
- Letting $\pi_{i,j} : (X \times X \times X) \to (X \times X)$ denote the projection onto i^{th} and j^{th} factors respectively,

$$\pi_{i,j}^* D_X(r) \wedge \pi_{j,k}^* D_X(s) \leq \pi_{i,k}^* D_X(r+s)$$

- Letting $r = \inf_i r_i$ for $r, r_i \in \mathbb{I}^{\text{op}}$, then $D_X(r) = \bigwedge_i D_X(r_i)$.
- Let $\pi_{X \times X} : (X \times Y \times X \times Y) \to (X \times X)$ and $\pi_{Y \times Y} : (X \times Y \times X \times Y) \to (Y \times Y)$ denote the projections preserving the ordering of the factors. Then

$$D_{X \times Y}(r) = (\pi_{X \times X})^* D_X(r) \wedge (\pi_{Y \times Y})^* D_Y(r)$$

Formalizing continuity

An inf- and 0-preserving increasing function $\epsilon : [0,1] \to [0,1]$ is a *modulus of continuity for* $f : X \to Y$ when for all $r \in [0,1]$

$$d_X(a,b) \leq r \implies d_Y(f(a), f(b)) \leq \epsilon(r).$$

Translating into our setting, we say an inf- and 0-preserving functor $\epsilon : \mathbb{I}^{\mathrm{op}} \to \mathbb{I}^{\mathrm{op}}$ is a modulus of continuity for $f : X \to Y$ when for all $r \in \mathbb{I}^{\mathrm{op}}$

$$D_X(r) \leq (f \times f)^* D_Y(\epsilon(r)).$$

Let $E \subseteq \mathrm{End}(\mathbb{I}^{\mathrm{op}})$ be the submonoid (under composition) of all such ϵ.

- Can vary E to allow only Lipschitz or 1-Lipschitz maps

Important real analysis properties of **pMet** follow *categorically* from our formulation.

Maps into $[0, 1]$

Consider $|\mathbb{I}| = [0, 1]$ with the obvious metric. Define $\mathcal{T}_{\mathbb{I}} : \mathbb{I}^{\mathrm{op}} \to \mathrm{Sub}\,|\mathbb{I}|$ by

$$\mathcal{T}_{\mathbb{I}}(r) = [0, r].$$

Lemma

A map $f : X \to |\mathbb{I}|$ is continuous w.r.t. $\epsilon \in E$ iff for all $r, s \in \mathbb{I}^{\mathrm{op}}$, we have

$$(\pi_1)^* f^* \mathcal{T}_{\mathbb{I}}(r) \wedge D_X(s) \leq (\pi_2)^* f^* \mathcal{T}_{\mathbb{I}}(r + \epsilon(s))$$

Continuous predicates

Definition

Given $X \in \mathbf{pMet}$ and $\epsilon \in E$, call $R : \mathbb{I}^{\text{op}} \to \operatorname{Sub} X$ an *ϵ-predicate on X* when

- For $r = \inf_i r_i$ in $\mathbb{I}^{\text{op}}$, $R(r) = \bigwedge_i R(r_i)$
- For all $r, s \in \mathbb{I}^{\text{op}}$,

$$(\pi_1)^* R(r) \wedge D_X(s) \leq (\pi_2)^* R(r + \epsilon(s))$$

Write $\operatorname{Sub}_\epsilon X \subseteq [\mathbb{I}^{\text{op}}, \operatorname{Sub} X]$ for the full subcategory on ϵ-predicates on X.

Proposition

For $f : X \to Y$ with modulus ϵ_f, and $R \in \operatorname{Sub}_\epsilon Y$, we have that $f^* R \in [\mathbb{I}^{\text{op}}, \operatorname{Sub} X]$ is an $(\epsilon \circ \epsilon_f)$-predicate.

Continuous predicate classifier

Recall $\mathcal{T}_{\mathbb{I}} : \mathbb{I}^{\text{op}} \to \text{Sub}\,|\mathbb{I}|$ is defined as

$$\mathcal{T}_{\mathbb{I}}(r) = [0, r]$$

so clearly $\mathcal{T}_{\mathbb{I}} \in \text{Sub}_{1_{\mathbb{I}^{\text{op}}}}\,|\mathbb{I}|$.

Theorem

- Given $f : X \to |\mathbb{I}|$ with modulus $\epsilon \in E$,

  $$R_f := f^* \mathcal{T}_{\mathbb{I}}$$

  is an ϵ-predicate on X.
- Given $R \in \text{Sub}_\epsilon X$, the function $f_R : X \to |\mathbb{I}|$ defined by

  $$f_R(x) = \inf\{r \in \mathbb{I}^{\text{op}} \mid x \in R(r)\}$$

  is a uniformly continuous map with modulus $\epsilon \in E$.

These operations are inverse to each other, and natural in X.

Sanity check

For any geometric category $\mathcal{C}$ (e.g. **Set**), we have $\mathbb{2}$-valued "metrics": given any $X \in \mathcal{C}$, set

$$D_X(0) = \text{diagonal}$$
$$D_X(1) = X \times X$$

Also let

$$\{1_{\mathbb{2}^{\mathrm{op}}}\} = E \subseteq \mathrm{End}(\mathbb{2}^{\mathrm{op}})$$

Then a "continuous predicate" on X is exactly just a subobject of X.

If we have some $\Omega \in \mathcal{C}$ and some given $\mathcal{T}_{\mathbb{2}} \in \mathrm{Sub}\,\Omega$, the analogous statement of the previous theorem (with Ω in place of $|\mathbb{I}|$) precisely means that Ω is a subobject classifier.

Quantification

Let $X \in \mathbf{pMet}$. The property of being an ϵ-predicate is closed under taking limits (meets) in $[\mathbb{I}^{\mathrm{op}}, \mathrm{Sub}\, X]$, so there is an adjunction

$$\mathrm{Sub}_\epsilon X \underset{i_\epsilon}{\overset{L_\epsilon}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\mathbb{I}^{\mathrm{op}}, \mathrm{Sub}\, X]$$

Given $f : X \to Y$ with modulus ϵ, we define the dashed functor in

$$\begin{array}{ccc} [\mathbb{I}^{\mathrm{op}}, \mathrm{Sub}\, X] & \underset{f^*}{\overset{\exists_f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & [\mathbb{I}^{\mathrm{op}}, \mathrm{Sub}\, Y] \\ L_{\epsilon\circ\epsilon_f} \big\downarrow \dashv \big\uparrow i_{\epsilon\circ\epsilon_f} & & L_\epsilon \big\downarrow \dashv \big\uparrow i_\epsilon \\ \mathrm{Sub}_{\epsilon\circ\epsilon_f} X & \dashrightarrow & \mathrm{Sub}_\epsilon Y \end{array}$$

as $L_\epsilon \exists_f i_{\epsilon\circ\epsilon_f}$, which we will also denote $\exists_f : \mathrm{Sub}_{\epsilon\circ\epsilon_f} X \to \mathrm{Sub}_\epsilon Y$ by abuse.

Proposition

$\exists_f : \mathrm{Sub}_{\epsilon\circ\epsilon_f} X \to \mathrm{Sub}_\epsilon Y$ is left adjoint to $f^* : \mathrm{Sub}_\epsilon Y \to \mathrm{Sub}_{\epsilon\circ\epsilon_f} X$.

Quantification

Recall that universal/existential quantification in continuous logic is sup/inf.

Proposition

For each $X, Y \in \mathbf{pMet}$ and each $R \in \mathrm{Sub}_\epsilon(Y \times X)$ with $\pi_X : Y \times X \to X$, the correspondence $R \mapsto f_R$ of the predicate classifier gives correspondences

$$\begin{aligned} \exists_{\pi_X} R \in \mathrm{Sub}_\epsilon X \quad &\mapsto \quad \inf_{y \in Y} f_R(y, -) : X \to |\mathbb{I}| \\ \text{For } Y \text{ inhabited: } \forall_{\pi_X} R \in \mathrm{Sub}_\epsilon R \quad &\mapsto \quad \sup_{y \in Y} f_R(y, -) : X \to |\mathbb{I}| \end{aligned}$$

In more general categories

Let $\mathcal{C}$ be (a variant of) a geometric category, and fix a submonoid $E \subseteq \mathrm{End}(\mathbb{I}^{\mathrm{op}})$.

Definition

We call $\mathcal{C}$ *metrizable* (w.r.t E) when it satisfies the conditions in the slide "formalizing the metric", plus:

For each morphism $f : X \to Y$ there is some $\epsilon \in E$ such that for all $r \in \mathbb{I}^{\mathrm{op}}$,

$$D_X(r) \leq (f \times f)^* D_Y(\epsilon(r))$$

A metrizable category has many of the features of the category **pMet**. In particular, the definition of ϵ-predicates makes sense, and their basic properties remain valid.

In more general categories

Also makes sense to ask for a *predicate classifier*:

Definition

A *predicate classifier* is given by an object $\Omega \in \mathcal{C}$ (and its metric D_Ω), along with a $1_{\mathbb{I}}$-predicate $\mathcal{C}_{\mathbb{I}} \in \mathrm{Sub}_{1_{\mathbb{I}}} \Omega$ such that:

For any $R \in \mathrm{Sub}_\epsilon X$, there is a unique $f : X \to \Omega$ such that $R = f^* \mathcal{T}_{\mathbb{I}}$, and moreover this f has modulus ϵ.

So **pMet** is a metrizable category which has a predicate classifier.

Presheaves of metric spaces

Let $\mathcal{C}$ be any small category.

Proposition

The category $\mathrm{MPSh}(\mathcal{C})$ of presheaves of metric spaces (with 1-Lipschitz maps between them) on $\mathcal{C}$ is metrizable w.r.t. $E = \{1_{\mathbb{I}}\}$.

(Nothing special going on above; just take the metric pointwise on $\mathcal{C}$.)

For convenience let us write $\mathrm{Sub}_{1_{\mathbb{I}^{\mathrm{op}}}} X$ as $\mathrm{Sub}_{\mathbb{I}} X$.

Theorem

$\mathrm{MPSh}(\mathcal{C})$ has a predicate classifier.

Presheaves of metric spaces

Need to give three things:

- A functor $\Omega : \mathcal{C}^{\mathrm{op}} \to \mathbf{pMet}_1$
- A specified predicate on Ω, i.e. $\mathcal{T}_{\mathbb{I}} \in \mathrm{Sub}_{\mathbb{I}}\, \Omega$

such that for any $X \in \mathrm{MPSh}(\mathcal{C})$, there is a natural correspondence

$$R \in \mathrm{Sub}_{\mathbb{I}}\, X \quad \mapsto \quad f_R : X \to \Omega$$

with $R = (f_R)^* \mathcal{T}_{\mathbb{I}}$.

For each $a \in \mathcal{C}$, the underlying set of $\Omega(a)$ is the set of functors $S : \mathbb{I}^{\mathrm{op}} \to \mathcal{S}_a$, where $\mathcal{S}_a$ is the poset of sieves on a, satisfying the property

$$\mathcal{S}(\inf_i r_i) = \bigwedge_i S(r_i)$$

Presheaves of metric spaces

For each $a \in \mathcal{C}$, we define a function $\nu : |\Omega(a)| \to [0,1]$ by

$$\nu_a(S) = \inf\{r \mid S(r) \text{ is the maximal sieve on } a\}$$

and a function $d_a^- : |\Omega(a)| \times |\Omega(a)| \longrightarrow [0,1]$ by

$$d_a^-(S_1, S_2) = |\nu_a(S_1) - \nu_a(S_2)|$$

We define $d_a : |\Omega(a)| \times |\Omega(a)| \longrightarrow [0,1]$ by

$$d_a(S_1, S_2) = \sup_{f:b \to a} d_b^-(|\Omega f|(S_1), |\Omega f|(S_2))$$

Define $\mathcal{T}_{\mathbb{I}} : \mathbb{I}^{\mathrm{op}} \to \mathrm{Sub}\,\Omega$ as

$$\mathcal{T}_{\mathbb{I}}(r) = \left(a \mapsto \nu_a^{-1}([0,r]) \subseteq \Omega(a)\right)$$