Relating the Effective Topos to Homotopy Type Theory

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Università di Genova

joint work with
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Overview of the talk

- Pseudo-equivalence relations and exact completions
- The effective topos $\mathcal{E}\!f\!f$
- The pseudo-equivalence relations in $\mathcal{A}sm$
- The cubical assemblies $\mathcal{A}sm^{\mathbb{C}^{\mathrm{op}}}$
- The embedding of $\mathcal{E}\!f\!f$ into a homotopy quotient of $\mathcal{A}sm^{\mathbb{C}^{\mathrm{op}}}$

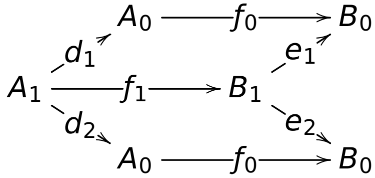
Dedicated to the memory of

Aurelio Carboni

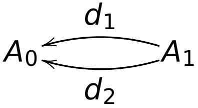
A category of pseudo-equivalence relations

$\mathscr{A}$: a category

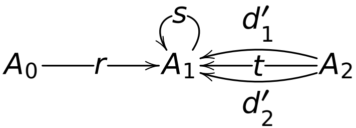
$\mathrm{Gph}(\mathscr{A}) \overset{\text{def}}{=}$



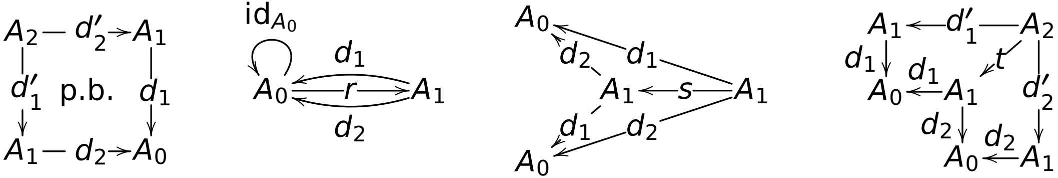
$\mathrm{PsER}(\mathscr{A}) \overset{\text{def}}{=}$ full subcategory of $\mathrm{Gph}(\mathscr{A})$ on the *ps.-equivalence relations*, *i.e.*



such that there are arrows



s.t.

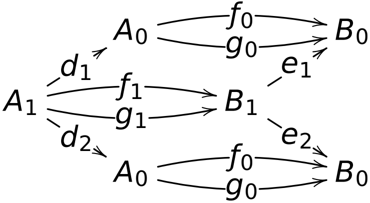


The exact completion of a category with finite limits

$\mathscr{A}$: a category with finite limits

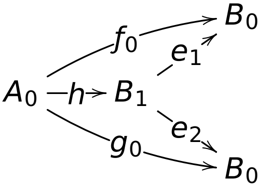
$\mathscr{A}_{ex}$ is a quotient category of PsER($\mathscr{A}$):

two arrows



of PsER($\mathscr{A}$) are *equivalent*

if there is a “half-homotopy”



in $\mathscr{A}$

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Graphs are internal presheaves

$\mathcal{A}$: a lextensive category

$\mathbb{G} \overset{\text{def}}{=}$ the internal category $\quad \mathrm{id}_b \circlearrowleft b \underset{\partial_2}{\overset{\partial_1}{\rightrightarrows}} a \circlearrowright \mathrm{id}_a$

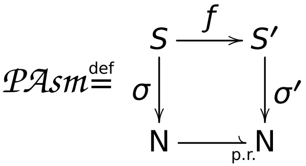
i.e. $\mathbb{G}_0 \overset{\text{def}}{=} \mathbb{T} + \mathbb{T}$ $\quad \mathbb{G}_1 \overset{\text{def}}{=} \mathbb{T} + \mathbb{T} + \mathbb{T} + \mathbb{T} \ldots$ $\quad$ *e.g.* $\mathbb{G}(b, a) = \underline{2}, \quad \mathbb{G}(a, b) = \underline{0}$

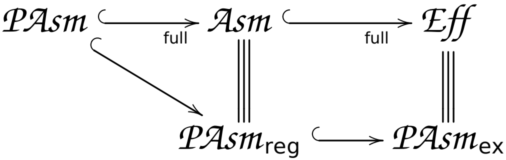
So $\quad \mathrm{PsER}(\mathcal{A}) \overset{}{\underset{\text{full}}{\hookrightarrow}} \mathrm{Gph}(\mathcal{A}) \equiv \mathcal{A}^{\mathbb{G}^{\mathrm{op}}}$

and, for instance, $\quad \mathrm{PsER}(\mathcal{PAsm}) \underset{\text{full}}{\hookrightarrow} \mathcal{PAsm}^{\mathbb{G}^{\mathrm{op}}}$

$$\mathrm{PsER}(\mathcal{PAsm}) \to \mathcal{PAsm}_{\mathrm{ex}} \equiv \mathcal{Eff}$$

Partitioned assemblies, assemblies, and the effective topos





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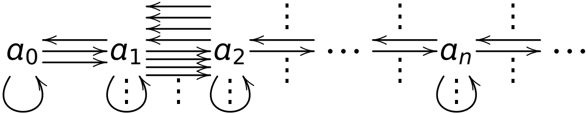
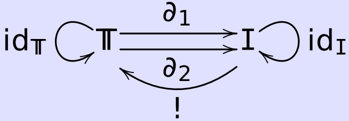
$\mathcal{A}$: a locally cartesian closed category with a natural number object $\mathbb{N}$

$\mathbb{C} \stackrel{\text{def}}{=}$ the internal category

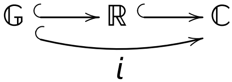


i.e. $\mathbb{C}_0 \stackrel{\text{def}}{=} \mathbb{N}$ $\quad \mathbb{C}_1 \stackrel{\text{def}}{=} \sum_{n,m:\mathbb{N}} \underline{n + 2^m} \ldots$ $\quad$ *e.g.* $\mathbb{C}(a_n, a_m) = \underline{n + 2^m}$

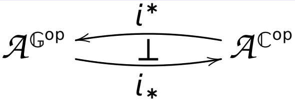
The internal category $\mathbb{C}$ in $\mathcal{A}$ is the free binary-product completion of the internal category $\mathbb{R}$



Note that



and



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$\mathcal{A}sm^{\mathbb{C}^{op}}$ is the quasitopos of *cubical assemblies*.

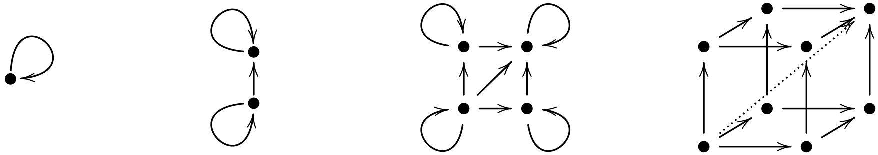
Note that

$$\mathcal{A}sm^{\mathbb{G}^{op}} \; \underset{i_*}{\overset{i_!}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{\;i^*\;}}}}} \; \mathcal{A}sm^{\mathbb{C}^{op}} \qquad (i_! \dashv i^* \dashv i_*)$$

$$G \longmapsto [n \mapsto \mathcal{A}sm^{\mathbb{G}^{op}}(\mathbb{I}^n, G)]$$

where $\mathbb{I}$ is an interval object which induces a cubical structure in $\mathcal{A}sm^{\mathbb{G}^{op}}$

$\mathbb{I}^0 = \mathbb{T}$ $\qquad \mathbb{I}^1 = \mathbb{I}$ $\qquad \mathbb{I}^2$ $\qquad \mathbb{I}^3$ $\qquad \dots$



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Kan fibrations of cubical assemblies

An *n-box* is $b\colon B \rightarrowtail \mathrm{I}^n$, $n > 0$, obtained by taking a decidable subobject $D \rightarrowtail \mathrm{I}^{n-1}$, and glueing $\mathrm{I} \times D \rightarrowtail \mathrm{I}^n$ to $D \rightarrowtail \mathrm{I}^{n-1} \rightarrowtail \mathrm{I}^n$.

A map $f\colon E \to F$ has the *right lifting property* with respect to $b\colon B \longrightarrow \mathrm{I}^n$ if, in every diagram

$$\begin{array}{ccc} B & \xrightarrow{p} & E \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{I}^n & \xrightarrow[q]{} & F \end{array}$$

there is a *diagonal filler*

$$\begin{array}{ccc} B & \xrightarrow{p} & E \\ {\scriptstyle b}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle f} \\ \mathrm{I}^n & \xrightarrow[q]{} & F \end{array}$$

A map $f\colon E \longrightarrow F$ is a *Kan fibration* if it has the r.l.p. with respect to all boxes.

$\mathrm{Kan}\left(\mathcal{A}sm^{\mathbb{C}^{\mathrm{op}}}\right) \overset{\mathrm{def}}{=}$ the full subcategory of $\mathcal{A}sm^{\mathbb{C}^{\mathrm{op}}}$ on the *(Kan) fibrant* objects, *i.e.* objects C such that $C \xrightarrow{!} \mathbb{T}$ is a Kan fibration.

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Ps.-equivalence relations of assemblies as cubical assemblies

$$\begin{array}{ccccc}
\mathcal{PAsm}^{\mathbb{G}^{\mathrm{op}}} & \underset{\text{full}}{\hookrightarrow} & \mathcal{Asm}^{\mathbb{G}^{\mathrm{op}}} & \xrightarrow{\quad i_* \quad} & \mathcal{Asm}^{\mathbb{C}^{\mathrm{op}}} \\
\uparrow{\scriptstyle \text{full}} & & \uparrow{\scriptstyle \text{full}} & & \uparrow{\scriptstyle \text{full}} \\
\mathrm{PsER}(\mathcal{PAsm}) & \underset{\text{full}}{\hookrightarrow} & \mathrm{PsER}(\mathcal{Asm}) & & \mathrm{Kan}\left(\mathcal{Asm}^{\mathbb{C}^{\mathrm{op}}}\right)
\end{array}$$

Ps.-equivalence relations of assemblies as cubical assemblies

$$
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\mathcal{PAsm}^{\mathbb{G}^{\mathrm{op}}} & \xhookrightarrow[\text{full}]{} & \mathcal{Asm}^{\mathbb{G}^{\mathrm{op}}} & \xrightarrow[i_*]{} & \mathcal{Asm}^{\mathbb{C}^{\mathrm{op}}} \\
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\end{array}
$$

Theorem. The functor $\mathrm{PsER}(\mathcal{Asm}) \hookrightarrow \mathcal{Asm}^{\mathbb{G}^{\mathrm{op}}} \xrightarrow[i_*]{} \mathcal{Asm}^{\mathbb{C}^{\mathrm{op}}}$

(i) is faithful

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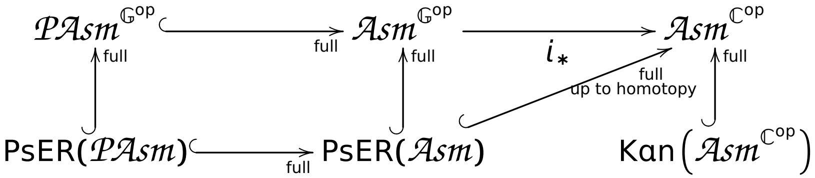
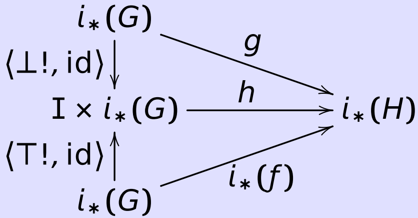
Ps.-equivalence relations of assemblies as cubical assemblies



Theorem. The functor $\mathrm{PsER}(\mathcal{Asm}) \hookrightarrow \mathcal{Asm}^{\mathbb{G}^{op}} \xrightarrow{i_*} \mathcal{Asm}^{\mathbb{C}^{op}}$

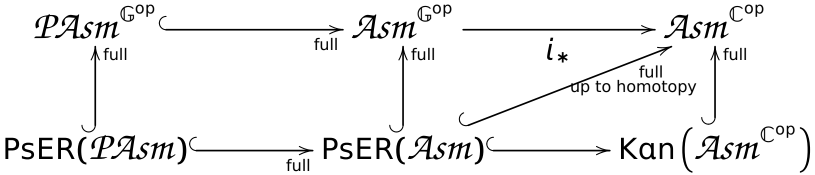
(i) is faithful

(ii) is full “up to homotopy”, in the sense that for every $g\colon i_*(G) \to i_*(H)$ in $\mathcal{Asm}^{\mathbb{C}^{op}}$ there is $f\colon G \to H$ in $\mathrm{PsER}(\mathcal{Asm})$ such that



commutes for some $h\colon \mathrm{I} \times i_*(G) \to i_*(H)$.

Ps.-equivalence relations of assemblies as cubical assemblies



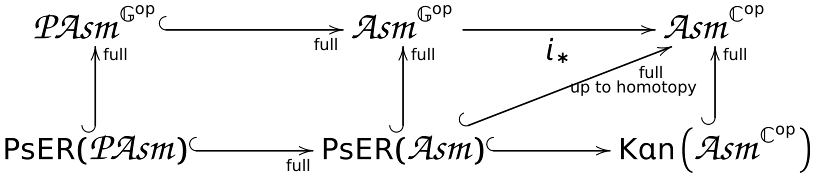
Theorem. The functor $\mathrm{PsER}(\mathcal{Asm}) \hookrightarrow \mathcal{Asm}^{\mathbb{G}^{op}} \xrightarrow{i_*} \mathcal{Asm}^{\mathbb{C}^{op}}$

(i) is faithful

(ii) is full “up to homotopy”

(iii) maps a ps.-equivalence relation G in PsER($\mathcal{Asm}$) to a Kan fibrant object in $\mathcal{Asm}^{\mathbb{C}^{op}}$.

Ps.-equivalence relations of assemblies as cubical assemblies



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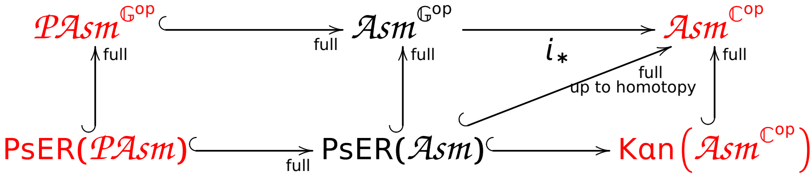
(i) is faithful

(ii) is full “up to homotopy”

(iii) maps a ps.-equivalence relation G in PsER($\mathcal{Asm}$) to a Kan fibrant object in $\mathcal{Asm}^{\mathbb{C}^{\mathrm{op}}}$.

Moreover, if the graph G is in $\mathcal{PAsm}$, and $i_*(G)$ is Kan fibrant, then G is a ps.-equivalence relation.

Ps.-equivalence relations of assemblies as cubical assemblies



Theorem. The functor $\mathrm{PsER}(\mathcal{Asm}) \hookrightarrow \mathcal{Asm}^{\mathbb{G}^{op}} \xrightarrow[i_*]{} \mathcal{Asm}^{\mathbb{C}^{op}}$

(i) is faithful

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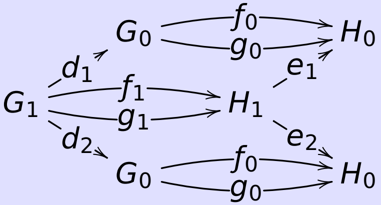
(iii) maps a ps.-equivalence relation G in $\mathrm{PsER}(\mathcal{Asm})$ to a Kan fibrant object in $\mathcal{Asm}^{\mathbb{C}^{op}}$.

Moreover, if the graph G is in $\mathcal{PAsm}$, and $i_*(G)$ is Kan fibrant, then G is a ps.-equivalence relation.

Homotopies for pseudo-equivalence relations

Theorem. Consider

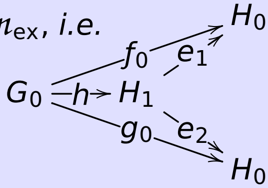


in PsER($\mathcal{A}sm$).

The following are equivalent:

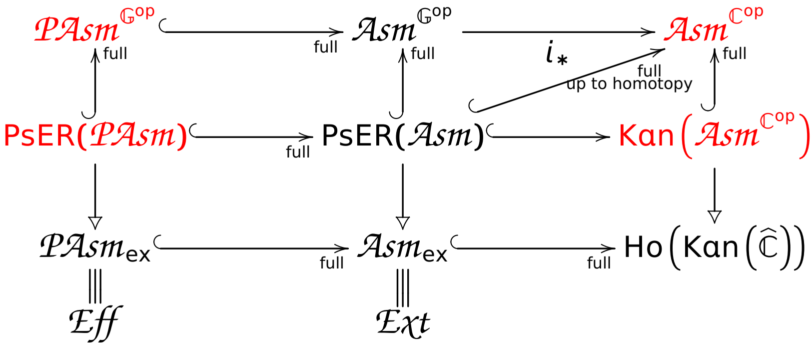
(i) f and g represent the same arrow in $\mathcal{A}sm_{\mathrm{ex}}$, *i.e.*

there is $h\colon G_0 \to H_1$ in $\mathcal{A}sm$ such that



(ii) the maps $i_*(G) \underset{i_*(g)}{\overset{i_*(f)}{\rightrightarrows}} i_*(H)$ are homotopically equivalent in $\widehat{\mathbb{C}}$.

Homotopies for pseudo-equivalence relations



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