Higher Modules and Directed Identity Types

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July 11, 2019

This work was supported by the Engineering and Physical Sciences Research Council.

A framework for formal higher category theory

- Virtual Double Categories
- Modules
- Globular Multicategories
- Higher Modules
- Weakening

Formal Category Theory

- Abstract setting for studying "category-like" structures
- Key notions of category theory can be defined once and for all

Virtual Double Categories

A virtual double category consists of a collection of:

- **objects** or 0-**types**

$$\bullet$$

$$A : 0\text{-Type}$$

- 0-**terms**

$$\begin{array}{c} A \\ \Big\downarrow{\scriptstyle f} \\ B \end{array}$$

$$x : A \vdash fx : B$$

Virtual Double Categories

- 1-**types**

$$A \xrightarrow{M} B$$

$$x : A, y : B \vdash M(x, y) : 1\text{-Type}(A, B)$$

Virtual Double Categories

- **1-terms**

$$\begin{array}{ccccc} A & \xrightarrow{M} & B & \xrightarrow{N} & C \\ {\scriptstyle f}\downarrow & & \Downarrow \phi & & \downarrow{\scriptstyle g} \\ D & & \xrightarrow[O]{\qquad\qquad\qquad} & & E \end{array}$$

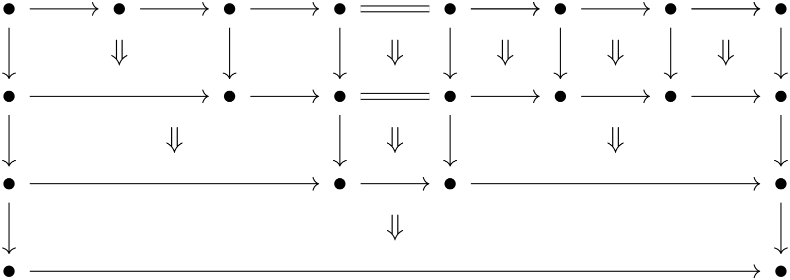
$$m : M(x, y), n : N(y, z) \vdash \phi(m, n) : O(fx, gz)$$

$$\begin{array}{ccc} A & = \!= \!= & A \\ {\scriptstyle f}\downarrow & \Downarrow \psi & \downarrow{\scriptstyle g} \\ D & \xrightarrow[O]{} & D \end{array}$$

$$a : A \vdash \psi(a) : O(fa, ga)$$

Virtual Double Categories

Terms have an associative and unital notion of composition

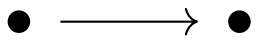


Example: Virtual Double Category of Categories

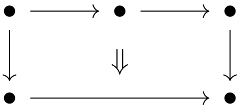
- 0-types are categories

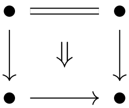
- 0-terms are functors

- 1-types are profunctors



- 1-terms are transformations between profunctors

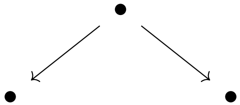




Example: Virtual Double Category of Spans

For any category $\mathcal{C}$ with pullbacks, there is a virtual double category $\mathsf{Span}(\mathcal{C})$ whose:

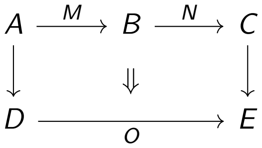
- 0-types are objects of $\mathcal{C}$
- 0-terms are arrows of $\mathcal{C}$
- 1-types are spans



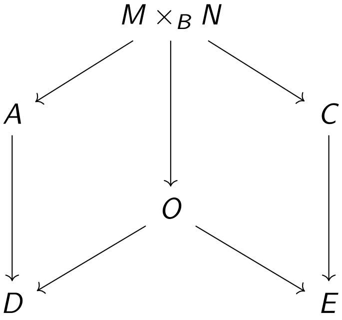
- 1-terms are transformations between spans.

Example: Virtual Double Category of Spans

- 1-terms are transformations between spans. A term



corresponds to a diagram



Identity Types

Typically for any 0-type A, there is a 1-type

$$A \xrightarrow{\mathcal{H}_A} A$$

which can be thought of as the **Hom-type** of A. This comes with a canonical **reflexivity term**

$$\begin{array}{ccc} A & = & A \\ \| & \Downarrow r_A & \| \\ A & \xrightarrow[\mathcal{H}_A]{} & A \end{array}$$

$$a : A \vdash r_A : \mathcal{H}_A(a, a)$$

Identity Types

Composition with

$$\begin{array}{ccc} A & = & A \\ \| & \Downarrow r_A & \| \\ A & \xrightarrow[\mathcal{H}_A]{} & A \end{array}$$

gives a bijection between terms of the following forms:

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{H}_A} & A \\ \| & \Downarrow & \| \\ B & \xrightarrow[M]{} & C \end{array} \qquad \begin{array}{ccc} A & = & A \\ \| & \Downarrow & \| \\ B & \xrightarrow[M]{} & C \end{array}$$

$$\frac{p : \mathcal{H}_A(x, y) \vdash \phi(p) : M(x, y)}{a : A \vdash \phi(r_a) : M(a, a)}$$

Identity Types

Composition with

$$\begin{array}{ccc} A & = & A \\ \| & \Downarrow r_A & \| \\ A & \xrightarrow[\mathcal{H}_A]{} & A \end{array}$$

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$$\frac{p : \mathcal{H}_A(x, y) \vdash \phi(p) : M(x, y)}{a : A \vdash \phi(r_a) : M(a, a)}$$

This is an abstract form of the **Yoneda Lemma**.

Identity Types

Composition with

$$\begin{array}{ccccc} A & = & A & \xrightarrow{M} & B \\ \| & \Downarrow r_A & \| & \Downarrow \mathrm{id}_M & \| \\ A & \xrightarrow[\mathcal{H}_A]{} & A & \xrightarrow[M]{} & B \end{array}$$

gives a bijection between terms of the following forms:

$$\begin{array}{ccccc} A & \xrightarrow{\mathcal{H}_A} & A & \xrightarrow{M} & B \\ {\scriptstyle f}\downarrow & & \Downarrow & & \downarrow{\scriptstyle g} \\ C & & \xrightarrow[N]{\hspace{6em}} & & D \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{M} & B \\ {\scriptstyle f}\downarrow & \Downarrow & \downarrow{\scriptstyle g} \\ C & \xrightarrow[N]{} & D \end{array}$$

$$\frac{p : \mathcal{H}_A(x, y), m : M(y, z) \vdash \phi(x, y, z, p, m) : N(fx, gz)}{y : A, m : M(y, z) \vdash \phi(y, y, z, r_y, m) : N(fy, gz)}$$

Identity Types

In fact $\mathcal{H}_A$ and r_A are characterised by such properties. We say that a virtual double category with this data has **identity types**.

Identity Types

- Let VDbl be the category of virtual double categories
- Let $\overline{\text{VDbl}}$ be the category of virtual double categories with identity types.
- The forgetful functor

$$U : \overline{\text{VDbl}} \to \text{VDbl}$$

  has both a left and a right adjoint.
- The right adjoint Mod is the monoids and modules construction.

Monoids and Modules

Given any virtual double category X, there is a virtual double $\mathrm{Mod}(X)$ such that:

- 0-types are monoids in X
  A monoid consists of a 0-type A, a 1-type $\mathcal{H}_A$ together with a unit

$$\begin{array}{ccc} A & = & A \\ \| & \Downarrow r_A & \| \\ A & \xrightarrow[\mathcal{H}_A]{} & A \end{array}$$

  and a multiplication

$$\begin{array}{ccccc} A & \xrightarrow{\mathcal{H}_A} & A & \xrightarrow{\mathcal{H}_A} & A \\ \| & & \Downarrow m_A & & \| \\ A & & \xrightarrow[\mathcal{H}_A]{} & & A \end{array}$$

  satisfying unit and associativity axioms.

Monoids and Modules

- 0-terms are monoid homomorphisms in X A monoid homomorphism $f : A \to B$ is a term

$$\begin{array}{ccc} A & \xrightarrow{\mathcal{H}_A} & A \\ \downarrow & \Downarrow f & \downarrow \\ B & \xrightarrow[\mathcal{H}_B]{} & B \end{array}$$

  compatible with the multiplication and unit terms of A and B.

Monoids and Modules

- 1-types are modules in X. A module $M : A \to B$ consists of a 1-type M together with left and right multiplication terms

$$
\begin{array}{ccccc}
A & \xrightarrow{\mathcal{H}_A} & A & \xrightarrow{M} & B \\
\Big\| & & \Downarrow \lambda_M & & \Big\| \\
A & & \xrightarrow[\mathcal{H}_A]{\qquad\qquad} & & A
\end{array}
\qquad
\begin{array}{ccccc}
A & \xrightarrow{M} & B & \xrightarrow{\mathcal{H}_B} & B \\
\Big\| & & \Downarrow \rho_M & & \Big\| \\
A & & \xrightarrow[\mathcal{H}_A]{\qquad\qquad} & & A
\end{array}
$$

compatible with the multiplication of A and B and each other.

Monoids and Modules

- 1-terms are module homomorphisms in X. A typical module homomorphism f is a term

$$\begin{array}{ccccc} A & \xrightarrow{M} & B & \xrightarrow{N} & C \\ \downarrow & & \Downarrow f & & \downarrow \\ D & & \xrightarrow[O]{\hspace{6em}} & & E \end{array}$$

  satisfying equivariance laws.

Equivariance Laws

For example

$$
\begin{array}{ccccccc}
A & \xrightarrow{M} & B & \xrightarrow{\mathcal{H}_B} & B & \xrightarrow{N} & C \\
\| & & \Downarrow \rho_M & & \| & = & \| \\
A & & \xrightarrow{M} & & B & \xrightarrow{N} & C \\
\downarrow & & & \Downarrow f & & & \downarrow \\
D & & & \xrightarrow[O]{} & & & E
\end{array}
\quad = \quad
\begin{array}{ccccccc}
A & \xrightarrow{M} & B & \xrightarrow{\mathcal{H}_B} & B & \xrightarrow{N} & C \\
\| & = & \| & & \Downarrow \lambda_N & & \| \\
A & \xrightarrow{M} & B & & \xrightarrow{N} & & C \\
\downarrow & & & \Downarrow f & & & \downarrow \\
D & & & \xrightarrow[O]{} & & & E
\end{array}
$$

Monoids and Modules

Many familiar types of "category-like" object are the result of applying the monoids and modules construction. For example:

- The virtual double category of categories internal to $\mathcal{C}$ is

$$\mathrm{Mod}(\mathrm{Span}(\mathcal{C}))$$

See

- T. Leinster. Higher Operads, Higher Categories
- G.S.H. Cruttwell and Michael A.Shulman. A unified framework for generalized multicategories

Formal Higher Category Theory

Virtual double categories are T-multicategories where T is the free category monad on 1-globular sets.

- Shapes of pasting diagrams of arrows in a category are parametrised by $T1$.
- The terms of a virtual double category are arrows sending such pasting diagrams of types to types.

$$\begin{array}{ccccc} & & X_1 & & \\ & \swarrow & & \searrow & \\ TX_0 & & & & X_0 \end{array}$$

Formal Higher Category Theory

Virtual double categories are T-multicategories where T is the free category monad on 1-globular sets.

- Shapes of pasting diagrams of arrows in a category are parametrised by $T1$.
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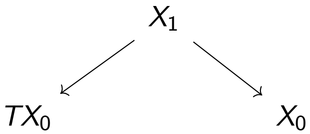
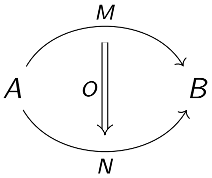


What about other T? In particular the free strict ω-category monad on globular sets

Globular Multicategories

A **globular multicategory** consists of a collection of:

- 0-**types**
- For each $n \geq 1$, n-**types**

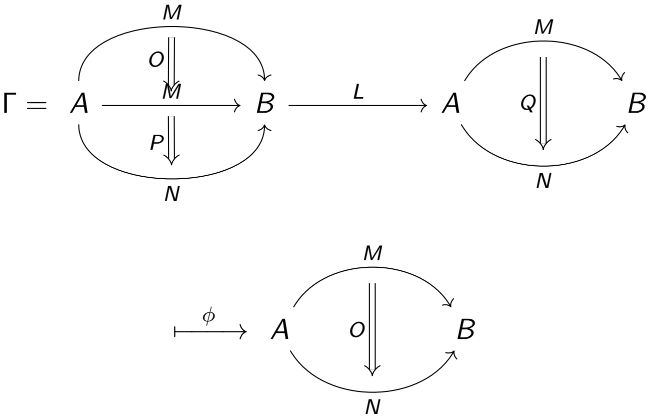


Suppose that we have parallel $(n-1)$-types A and B. Given $M(u, v) : n\text{-Type}(A, B)$ and $N(u, v) : n\text{-Type}(A, B)$, we have

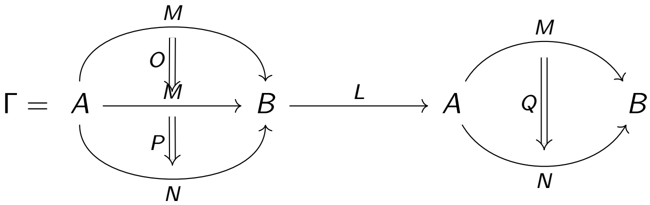
$$x : M(u, v), y : N(u, v) \vdash O(x, y) : (n+1)\text{-Type}(M, N)$$

Globular Multicategories

- *n*-**terms** sending a pasting diagram of types to an *n*-type.



Globular Multicategories



$$
\begin{aligned}
\Gamma(0) = \quad & [a : A, b : B, a' : A, b' : B] \\
\Gamma(1) = \quad & [m : M(a,b), m' : M(a,b), n : N(a,b), \\
& l : L(b,a'), m' : M(a',b'), n' : N(a',b')] \\
\Gamma(2) = \quad & [o : O(m,n), p : P(m,n'), q : Q(m',n')]
\end{aligned}
$$

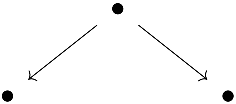
We have

$$\Gamma \vdash \phi(l, o, p, q) : O(a, b')$$

Example: Globular Multicategory of Spans

For any category $\mathcal{C}$ with pullbacks, there is a globular multicategory Span($\mathcal{C}$) whose:

- 0-types are objects of $\mathcal{C}$
- 1-types are spans

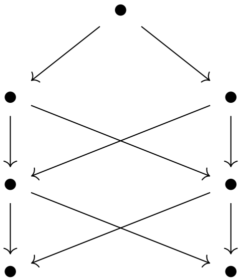


- 2-types are spans between spans. (That is 2-spans.)
- 3-types are spans between 2-spans (That is 3-spans).

Example: Globular Multicategory of Spans

For any category $\mathcal{C}$ with pullbacks, there is a globular multicategory Span($\mathcal{C}$) whose:

- 0-types are objects of $\mathcal{C}$
- 1-types are spans
- 2-types are spans between spans (or 2-spans)
- 3-types are spans between 2-spans (That is 3-spans). That is a diagram



Example: Globular Multicategories of Spans

For any category $\mathcal{C}$ with pullbacks, there is a globular multicategory Span($\mathcal{C}$) whose:

- 0-types are sets
- 0-terms are functions
- 1-types are spans
- 2-types are spans between spans (or 2-spans)
- 3-types are spans between 2-spans (That is 3-spans).
- etc.
- Terms are transformations from a pullback of spans to a span.

Globular Multicategories associated to Type Theories

- There is a globular multicategory associated to any model of dependent type theory
- Types, contexts and terms correspond to the obvious things in the type theory.
- See Benno van den Berg and Richard Garner. Types are weak ω-groupoids

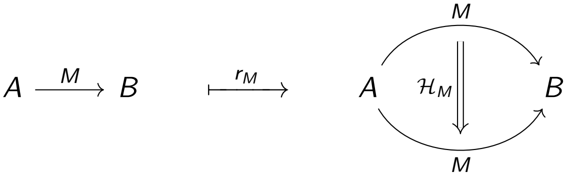
Globular Multicategories associated to Type Theories

- There is a globular multicategory associated to any model of dependent type theory
- Types, contexts and terms correspond to the obvious things in the type theory.
- See Benno van den Berg and Richard Garner. Types are weak ω-groupoids

When we have identity types, what structure does this globular multicategory have?

Globular Multicategories with Strict Identity Types

- For each n-type M, we require an identity $(n+1)$ type $\mathcal{H}_M$ with a reflexivity term $r : M \to \mathcal{H}_M$.



- Composition with reflexivity terms gives bijective correspondences which "add and remove identity" types

Globular Multicategories with Strict Identity Types

- The forgetful functor

$$U : \overline{\mathsf{GlobMult}} \to \mathsf{GlobMult}$$

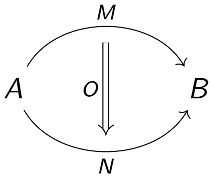
  has both a left and a right adjoint.
- The right adjoint Mod is the strict higher modules construction.

Higher Modules

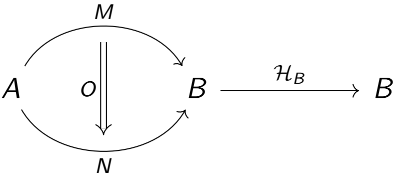
In general, n-modules can be acted on by their k-dimensional source and target modules for any $k < n$.

Higher Modules

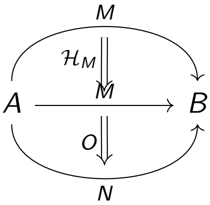
Given a 2-module O, depicted



there are actions whose sources are



and

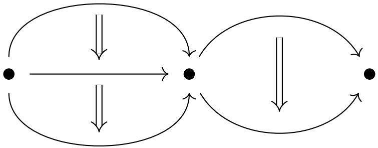


Higher Module Homomorphisms

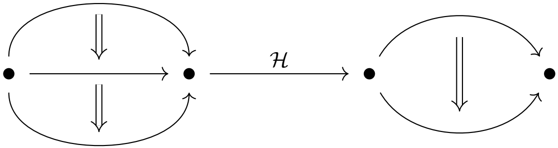
Given a homomorphism f with source Γ, there is an equivariance law for each place in Γ that an identity type can be added.

Higher Module Homomorphisms

Given a homomorphism f with source



there are two ways of building terms with source



using either left or right actions.

Globular multicategory of strict ω-categories

Applying this construction to Span(Set) we obtain a globular multicategory whose

- 0-types are strict ω-categories,
- 1-types are profunctors
- 2-types are profunctors between profunctors
- etc.
- 0-terms are strict ω-functors,
- Higher terms are transformations between profunctors

Weakening

- Let

$$U : \overline{T}\text{-}\mathrm{Mult} \to \overline{T}\text{-}\mathrm{Mult}$$

be the functor which forgets strict identity types. Let

$$F : T\text{-}\mathrm{Mult} \to \overline{T}\text{-}\mathrm{Mult}$$

be its left adjoint. Let u be a generic type (or term). We have

$$\cfrac{\cfrac{u \longrightarrow U\,\mathrm{Mod}(X)}{Fu \longrightarrow \mathrm{Mod}(X)}}{UFu \longrightarrow X}$$

Weakening

- The boundary inclusions of the shapes of globular multicategory cells, induce a weak factorization system.
- A weak map of globular multicategories is a strict map from a **cofibrant replacement**

$$QX \longrightarrow Y$$

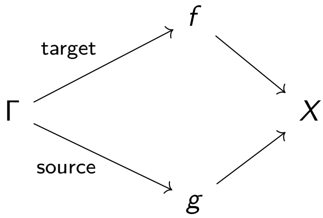
- Thus, we define a weak n-module (or homomorphism) to be a map

$$QUFu \longrightarrow X$$

- Weak 0-modules are precisely Batanin-Leinster ω-categories. See Richard Garner. A homotopy-theoretic universal property of Leinster's operad for weak ω-categories

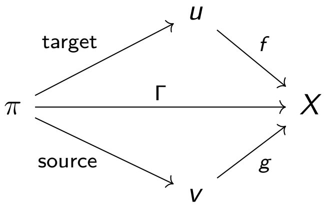
Composition of weak higher module homomorphisms

- A pair of composable terms in a globular multicategory is the same as a diagram

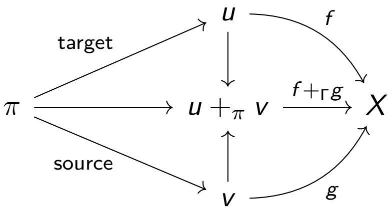


Composition of weak higher module homomorphisms

- Let Γ be a context in X with shape π and let $u : \Delta \to \Gamma, v : \Gamma \to A$ be a composable pair in X. Then we have a commutative diagram

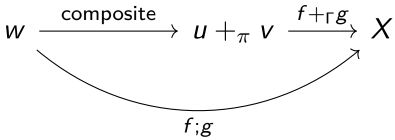


- Hence, we have a diagram

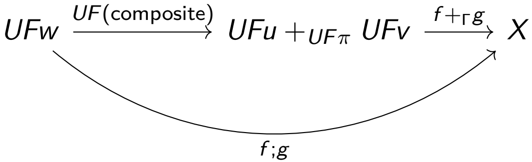


Composition of weak higher module homomorphisms

- Let w be the shape of u, v. Then $f;g$ is defined by the following commutative diagram:



- Since UF is cocontinuous, composition of strict homomorphisms defined by the following commutative diagram:



Composition of weak higher module homomorphisms

- We would like a diagram

$$
\begin{array}{c}
QUFw \xrightarrow{QUF(\text{composite})} QUFu +_{QUF\pi} QUFv \xrightarrow{f +_{\Gamma} g} X \\
QUFw \xrightarrow{f;g} X
\end{array}
$$

  but Q is not cocontinuous.
- However $QUFu +_{QUF\pi} QUFv$ is still cofibrant. This allows us to construct a well-behaved composition map

$$QUFw \longrightarrow QUFu +_{QUF\pi} QUFv$$

Weak Modules

Applying this construction to Span(Set) we obtain notions of

- Weak ω-categories, profunctors, profunctors between profunctors, etc.
- Weak transformations between profunctors
- Composition of these terms

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We can use data to construct an ω-category of ω-categories

Future Work

- Semi-strictness results and comparison to dependent type theory.
- Develop higher category theory and higher categorical logic.

Thanks

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