A General Framework for the Semantics of Type Theory

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CwF-semantics of Type Theory

Semantics of type theories based on *categories with families* (CwF) (Dybjer 1996).

- Martin-Löf type theory
- Homotopy type theory
- Homotopy type system (Voevodsky 2013) and two-level type theory (Annenkov, Capriotti, and Kraus 2017)
- Cubical type theory (Cohen et al. 2018)

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Goal

To define a general notion of a "type theory" to unify the CwF-semantics of various type theories.

Outline

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Natural Models

An alternative definition of CwF.

Definition (Awodey 2018)

A *natural model* consists of...

- a category $\mathcal{S}$ (with a terminal object);
- a map $p : E \to U$ of presheaves over $\mathcal{S}$

such that p is representable: for any object $\Gamma \in \mathcal{S}$ and element $A \in U(\Gamma)$, the presheaf A^*E defined by the pullback

$$\begin{array}{ccc} A^*E & \longrightarrow & E \\ \big\downarrow & \lrcorner & \big\downarrow p \\ よ\Gamma & \xrightarrow[A]{} & U \end{array}$$

is representable, where よ is the Yoneda embedding.

CwF vs Natural Model

The representable map $p : E \to U$ models *context comprehension*:

$$
\begin{array}{ccc}
よ\{A\} & \xrightarrow{\delta_A} & E \\
{\scriptstyle \pi_A}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
よ\Gamma & \xrightarrow[A]{} & U
\end{array}
\qquad よ\{A\} \cong A^* E
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Proposition (Awodey 2018)

CwFs $\simeq$ natural models.

Modeling Type Formers

Dependent function types (Π-types) are modeled by a pullback

$$\begin{array}{ccc} P_p E & \xrightarrow{\lambda} & E \\ {\scriptstyle P_p p}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ P_p U & \xrightarrow[\Pi]{} & U \end{array}$$

where $P_p : [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}] \to [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}]$ is the functor

$$[\mathcal{S}^{\mathsf{op}}, \mathbf{Set}] \xrightarrow{(-\times E)} [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}]/E \xrightarrow{p_*} [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}]/U \xrightarrow{\mathsf{dom}} [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}]$$

and p_* is the pushforward along p, i.e. the right adjoint of the pullback p^*.

Summary on Natural Models

An (extended) natural model consists of...

- a category $\mathcal{S}$ (with a terminal object);
- some presheaves $U, E, \ldots$ over $\mathcal{S}$;
- some representable maps $p : E \to U, \ldots$;
- some maps $X \to Y$ of presheaves over $\mathcal{S}$ where X and Y are built up from $U, E, \ldots, p, \ldots$ using finite limits and pushforwards along the representable maps $p, \ldots$.

Outline

Representable Map Categories

Definition

A *representable map category* is a category $\mathcal{A}$ equipped with a class of arrows called representable arrows satisfying the following:

- $\mathcal{A}$ has finite limits;
- identity arrows are representable and representable arrows are closed under composition;
- representable arrows are stable under pullbacks;
- the pushforward $f_* : \mathcal{A}/X \to \mathcal{A}/Y$ along a representable arrow $f : X \to Y$ exists.

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Definition

A *representable map functor* $F : \mathcal{A} \to \mathcal{B}$ between representable map categories is a functor $F : \mathcal{A} \to \mathcal{B}$ preserving all structures: representable arrows; finite limits; pushforwards along representable arrows.

Type Theories

Definition

A *type theory* is a (small) representable map category $\mathbb{T}$.

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Definition

A *model* of a type theory $\mathbb{T}$ consists of...

- a category $\mathcal{S}$ with a terminal object;
- a representable map functor $(-)^{\mathcal{S}} : \mathbb{T} \to [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}]$.

Examples of Type Theories

Proposition

Representable map categories have some "free" constructions (cf. LCCCs and Martin-Löf type theories (Seely 1984)).

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Example

If $\mathbb{T}$ is freely generated by a single representable arrow $p : E \to U$, a model of $\mathbb{T}$ consists of...

- a category $\mathcal{S}$ with a terminal object;
- a representable map $p^{\mathcal{S}} : E^{\mathcal{S}} \to U^{\mathcal{S}}$ of presheaves over $\mathcal{S}$

i.e. a natural model.

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Main Results

Let $\mathbb{T}$ be a type theory.

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Theorem

There is a "theory-model correspondence": we define a (locally discrete) 2-category $\mathbf{Th}_{\mathbb{T}}$ of $\mathbb{T}$-theories and establish a bi-adjunction

$$\mathbf{Mod}_{\mathbb{T}} \underset{\mathsf{L}_{\mathbb{T}}}{\overset{\mathsf{M}_{\mathbb{T}}}{\underset{\perp}{\rightleftarrows}}} \mathbf{Th}_{\mathbb{T}}.$$

The Bi-initial Model

For a type theory $\mathbb{T}$, we define a model $\mathcal{J}(\mathbb{T})$ of $\mathbb{T}$:

- the base category is the full subcategory of $\mathbb{T}$ consisting of those $\Gamma \in \mathbb{T}$ such that the arrow $\Gamma \to 1$ is representable;
- we define $(-)^{\mathcal{J}(\mathbb{T})}$ to be the composite

$$\mathbb{T} \xrightarrow{よ} [\mathbb{T}^{\mathsf{op}}, \mathbf{Set}] \to [\mathcal{J}(\mathbb{T})^{\mathsf{op}}, \mathbf{Set}].$$

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Given a model $\mathcal{S}$ of $\mathbb{T}$, we have a functor

$$\begin{array}{ccc} \mathcal{J}(\mathbb{T}) & \xrightarrow{\mathsf{F}} & \mathcal{S} \\ \big\downarrow & \cong & \big\downarrow \text{よ} \\ \mathbb{T} & \xrightarrow[(-)^{\mathcal{S}}]{} & [\mathcal{S}^{\mathsf{op}}, \mathbf{Set}] \end{array}$$

and F can be extended to a morphism of models of $\mathbb{T}$.

Internal Languages

Definition

We define a 2-functor $\mathsf{L}_{\mathbb{T}} : \mathbf{Mod}_{\mathbb{T}} \to \mathbf{Cart}(\mathbb{T}, \mathbf{Set})$ by $\mathsf{L}_{\mathbb{T}}\mathcal{S}(A) = A^{\mathcal{S}}(1)$, where $\mathbf{Cart}(\mathbb{T}, \mathbf{Set})$ is the category of functors $\mathbb{T} \to \mathbf{Set}$ preserving finite limits.

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Theorem

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Theorem

$\mathsf{L}_{\mathbb{T}} : \mathbf{Mod}_{\mathbb{T}} \to \mathbf{Cart}(\mathbb{T}, \mathbf{Set})$ *has a left bi-adjoint with invertible unit.*

$$\mathbf{Th}_{\mathbb{T}} := \mathbf{Cart}(\mathbb{T}, \mathbf{Set})$$

(Cf. algebraic approaches to dependent type theory (Isaev 2018; Garner 2015; Voevodsky 2014))

Conclusion

- A type theory is a representable map category.
- Every type theory has a bi-initial model.
- There is a theory-model correspondence.

Future Directions:

- Application: canonicity by gluing representable map categories?
- What can we say about the 2-categoty $\mathbf{Mod}_{\mathbb{T}}$?
- Better presentations of the category $\mathbf{Th}_{\mathbb{T}}$?
- Variations: internal type theories? $(\infty, 1)$-type theories?

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Why is it a Theory?

In algebraic approaches to dependent type theory (Isaev 2018; Garner 2015; Voevodsky 2014), a *theory* is a diagram in $\mathbf{Set}$ which looks like

$$
\begin{array}{ccccccc}
 & & E_0 & & E_1 & & E_2 & & \dots \\
 & \swarrow & & \swarrow & & \swarrow & & & \\
U_0 & \longleftarrow & U_1 & \longleftarrow & U_2 & \longleftarrow & \dots & &
\end{array}
$$

where

U_n set of types with n variables;

E_n set of terms with n variables.

Why is it a Theory?

If $\mathbb{T}$ has a representable arrow $p : E \to U$, then $\mathbb{T}$ contains a diagram

$$
\begin{array}{ccccccc}
 & & P_p^0 E & & P_p^1 E & & P_p^2 E & & \dots \\
 & {\scriptstyle P_p^0 p} \swarrow & & {\scriptstyle P_p^1 p} \swarrow & & {\scriptstyle P_p^2 p} \swarrow & & & \\
P_p^0 U & \longleftarrow & P_p^1 U & \longleftarrow & P_p^2 U & \longleftarrow & \dots & &
\end{array}
$$

where $P_p X = p_*(X \times E)$.