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Univalence and completeness of Segal objects

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Outline

Type theoretic model categories

Definition (sort of)

A *type theoretic model category* $\mathbb{M}$ is a model category such that its associated category $\mathbb{C} := \mathbb{M}^f$ of fibrant objects is a type theoretic fibration category.

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Recall

1. Complete Segal spaces are Reedy fibrant simplicial objects in $(\mathbf{S}, \mathrm{Kan})$ satisfying *the Segal conditions* and *the completeness condition*.

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The Quillen model structure $(\mathbf{S}, \mathrm{Kan})$.

Recall

2. There is a model structure $(s\mathbf{S}, \mathrm{CS})$ whose fibrant objects are the complete Segal spaces.
   $\rightsquigarrow$ Classical model for $(\infty, 1)$-category theory.

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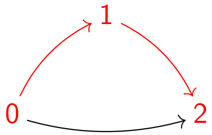

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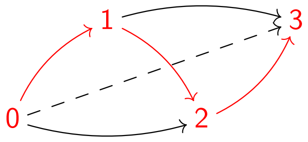

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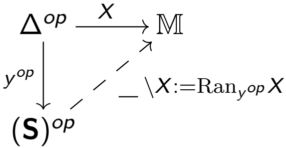

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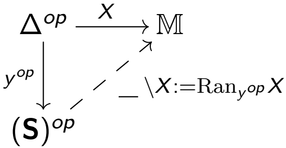

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Definition
The *n-th Segal map* associated to a simplicial object X in $\mathbb{M}$ is the map

$$\iota_n \setminus X \colon \Delta^n \setminus X \to I_n \setminus X.$$

Let $X \in s\mathbb{C}$ be a simplicial object in $\mathbb{C}$.

$$\begin{array}{ccc} \Delta^{op} & \xrightarrow{X} & \mathbb{M} \\ {\scriptstyle y^{op}}\downarrow & \nearrow_{\_ \backslash X := \mathrm{Ran}_{y^{op}} X} & \\ (\mathbf{S})^{op} & & \end{array}$$

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The *n-th Segal map* associated to a simplicial object X in $\mathbb{M}$ is the map

$$\xi_n \colon X_n \to (X_{1/X_0})^n_S.$$

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Let $X \in s\mathbb{C}$ be a simplicial object in $\mathbb{C}$.

1. X is *sufficiently fibrant* if both the 2-Segal map

$$\xi_2 \colon X_2 \to X_1 \times_{X_0} X_1$$

   and the boundary map

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2. Let X be sufficiently fibrant. We say that X is a *Segal object* (*strict Segal object*) if the associated Segal maps

$$\xi_n \colon X_n \to (X_{1/X_0})^n_S$$

   are homotopy equivalences (isomorphisms) in $\mathbb{C}$.

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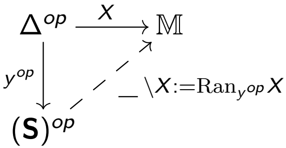
$\rightsquigarrow$ $N\mathrm{Fun}(p)$ is a strict Segal object.

Proposition

Let $p\colon E \twoheadrightarrow B$ be a fibration in $\mathbb{C}$. Then p is a univalent fibration in $\mathbb{C}$ if and only if the Segal object $N\mathrm{Fun}(p)$ is univalent.

Rezk Completeness

Let $X \in s\mathbb{C}$ be a simplicial object in $\mathbb{C}$. Recall the Kan extension



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Definition

A Reedy fibrant Segal object X is *complete* if the functor

$$\_ \setminus X \colon (\mathbf{S}, \mathrm{QCat})^{op} \to \mathbb{M}$$

is a right Quillen functor.

A map $\mathcal{C} \to \mathcal{D}$ between quasi-categories is a quasi-fibration if and only if it has the right lifting property against

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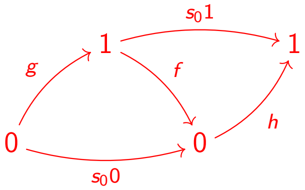

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Definition
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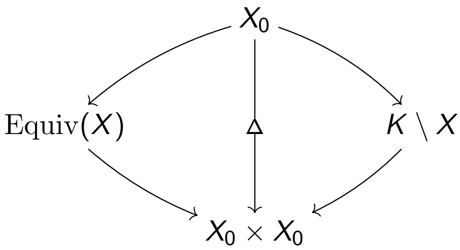
For every Segal object $X \in s\mathbb{C}$, we have



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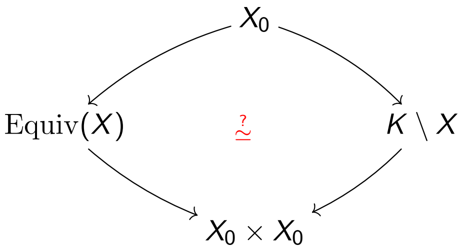

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Theorem

Let X be a Segal object in $\mathbb{C}$. Then X is univalent if and only if its Reedy fibrant replacement $\mathbb{R}X$ is complete.

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Corollary

Let Xp be a fibration in $\mathbb{C}$. Then p is a univalent fibration if and only if the Segal object $\mathbb{R}N(\mathrm{Fun}(p))$ is complete.

Univalent and Rezk completion

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Definition

We say that $p\colon E \twoheadrightarrow B$ is *small* if it arises as the homotopy pullback of π along some map $B \to U$.

Definition

Let $p\colon E \twoheadrightarrow B$ be a small fibration in $\mathbb{C}$. We say that a homotopy cartesian square

$$\begin{array}{ccc} E & \longrightarrow & u(E) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle u(p)} \\ B & \underset{\iota}{\longrightarrow} & u(B) \end{array}$$

is a *univalent completion* of p if the fibration $u(p) \in \mathbb{C}$ is small and univalent, and the map $\iota\colon B \to u(B)$ is a (-1)-connected cofibration.

Proposition

For every fibration $p\colon E \twoheadrightarrow B$ in $\mathbb{C}$ there is a univalent completion

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Proof.

$$\begin{array}{ccc} E & \longrightarrow & \tilde{U} \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow[b]{} & U \end{array}$$

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□

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Given a map $f\colon X \to Y$ of Segal objects in $\mathbb{C}$, consider

$$
\begin{array}{ccc}
\mathrm{Eq} f & \longrightarrow & \mathrm{Equiv}\, Y \\
\big\downarrow & \lrcorner & \big\downarrow \\
f \downarrow Y & \longrightarrow & Y_1 \\
\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle (s,t)} \\
X_0 \times Y_0 & \xrightarrow[f_0 \times 1]{} & Y_0 \times Y_0
\end{array}
$$

Definition

Let $f\colon X \to Y$ be a map between Segal objects in $\mathbb{M}$. We say that

1. f is *fully faithful* if the natural map $X_1 \to (f_0 \times f_0)^* Y_1$ over $X_0 \times X_0$ is a weak equivalence.
2. f is *essentially surjective* if the fibration $(\mathrm{Eq} f)_{-1} \twoheadrightarrow Y_0$ is acyclic.
3. f is a *DK-equivalence* if it is fully faithful and essentially surjective.

Theorem

For every fibration $p\colon E \twoheadrightarrow B$, the univalent completion

$$\begin{array}{ccc} E & \longrightarrow & u(E) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle u(p)} \\ B & \xrightarrow[\iota]{} & u(B) \end{array}$$

induces a DK-equivalence

$$\mathbb{R}N(\iota)\colon \mathbb{R}N(p) \to \mathbb{R}N(u(p))$$

from the Segal object $\mathbb{R}N(p)$ to the complete Segal object $\mathbb{R}N(u(p))$ in $\mathbb{C}$.

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- Discuss Rezk completion in the sense of Ahrens, Kapulkin and Shulman.

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- This suggests that univalent fibrations might be the fibrant objects in some fibration category?

Thank you!

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