New model structures on simplicial sets

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CT 2019 — Edinburgh

Outline

- 0-simplices: $\bullet$

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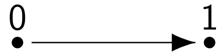
- 1-simplices: 

Category Theory in Simplicial Sets

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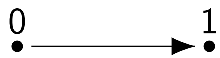
- 1-simplices: 

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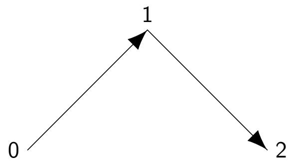

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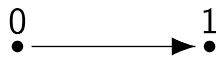
- 1-simplices: 

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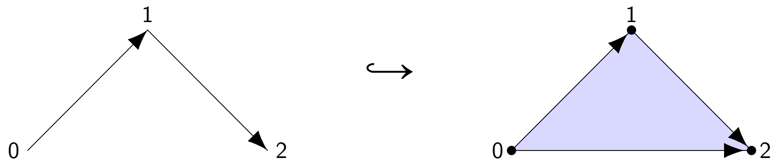

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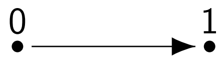
- 1-simplices: 

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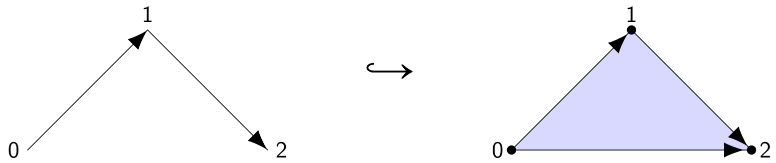

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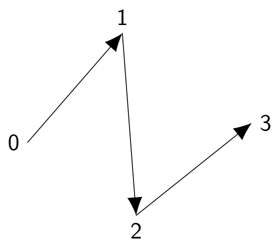

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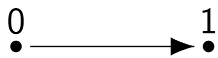
- 1-simplices: 

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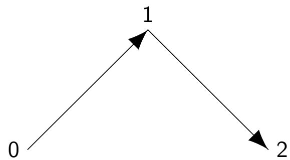

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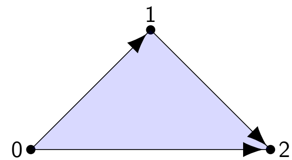

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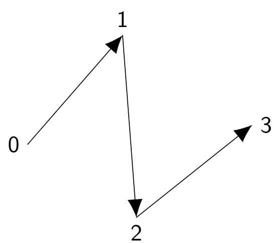

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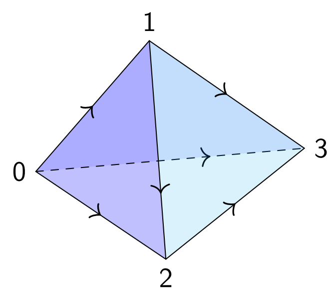

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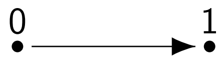
- 1-simplices: 

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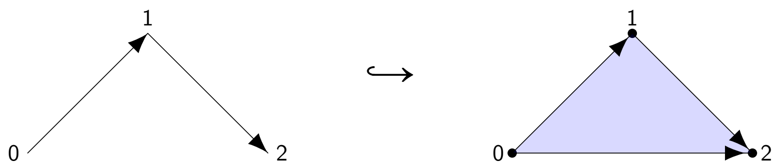

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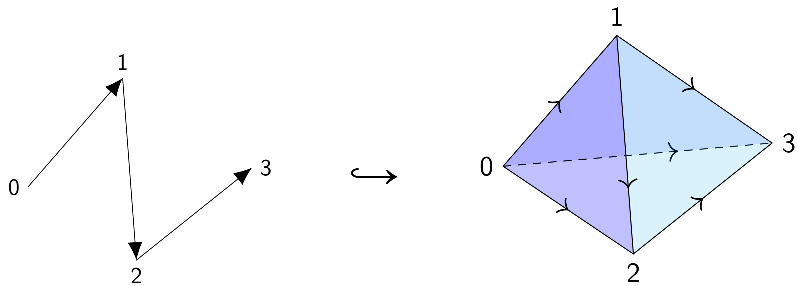

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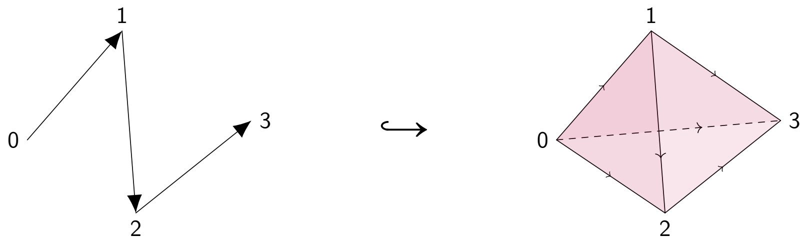
$$\Delta[n] = N(\bullet \xrightarrow{g_1} \dots \xrightarrow{g_n} \bullet) \qquad J := N\left(\bullet \underset{g}{\overset{g^{-1}}{\leftrightarrows}} \bullet\right)$$

Categories are “Strict”

The inclusion $\mathcal{S}p[n] \hookrightarrow \Delta[n]$



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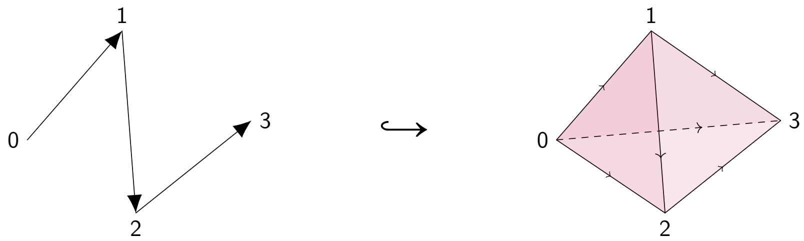

is called a *spine extension*.

Unique Lifting = "Strict"

Categories are simplicial sets with unique spine extensions.

$X \cong N(C)$

(for some C)

$\iff$

$$\begin{array}{ccc} \mathcal{S}p[n] & \longrightarrow & X \\ \big\downarrow & \nearrow_{\exists!} & \\ \Delta[n] & & \end{array}$$

$n \geqslant 2$

Categories are “1-Segal Sets”

Unique lifting condition for categories is 1-dimensional:

$\mathcal{S}p[n]$ is composed of 1-simplices.

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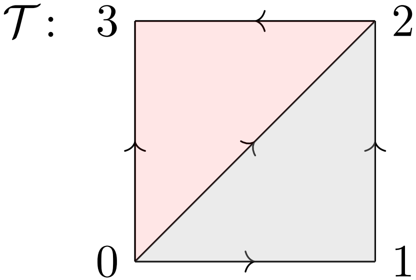
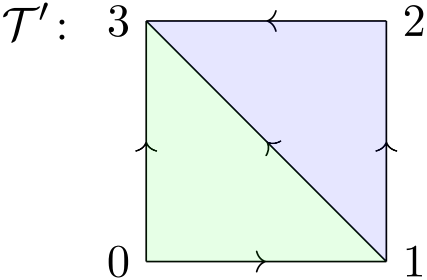
- More general than categories
- Unique "2-dimensional spine extensions"
  - ↳ still "strict"

Triangulations of the square:





2-Segal Sets

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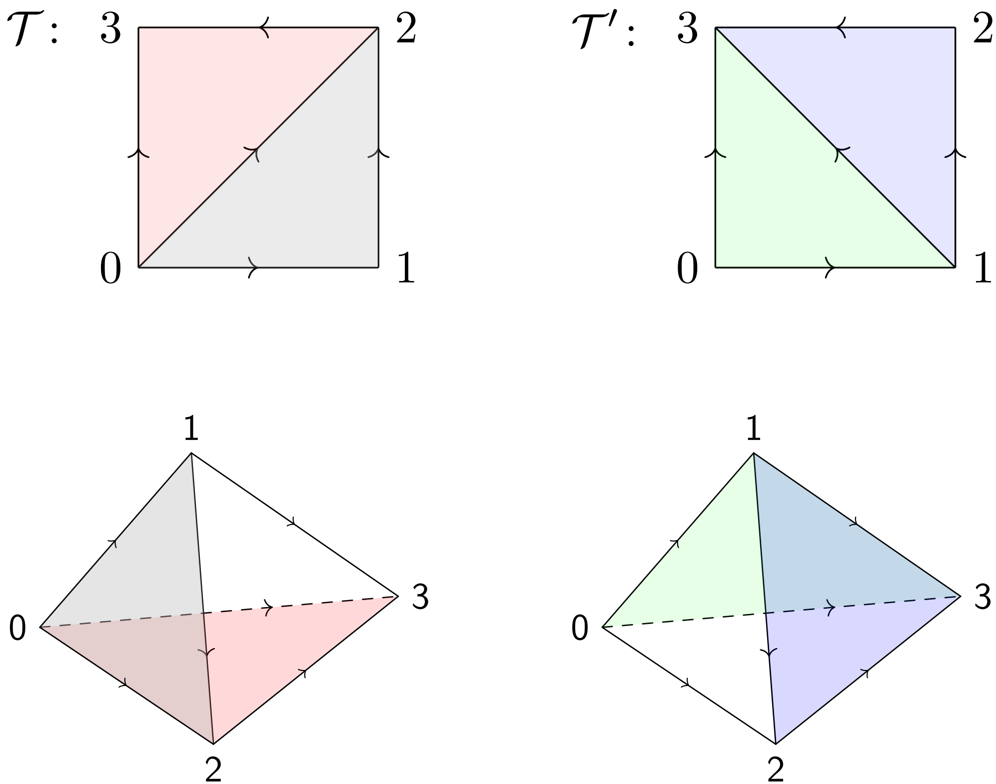
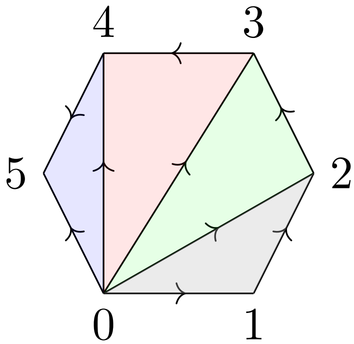
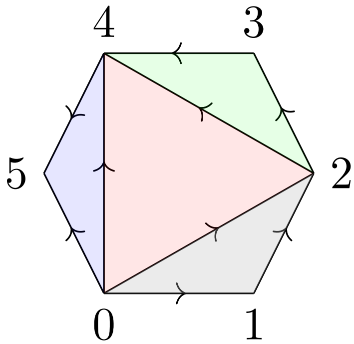


Triangulations of the hexagon:

$\mathcal{T}$:



$\mathcal{T}'$:



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Intuition

Think of the inclusions $\mathcal{T} \hookrightarrow \Delta[n]$ as "2-dimensional spine extensions."

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Definition

A *2-Segal set* is a simplicial set X with a unique lifting condition:

X is 2-Segal $\iff$

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & X \\ \big\downarrow & \overset{\exists !}{\nearrow} & \\ \Delta[n] & & \end{array}$$

$n \geqslant 3$

Examples

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Another Perspective

2-Segal sets are equivalent to *multivalued categories*, where composition is not always unique or defined, but is associative.

Homotopical/∞/Non-Strict Versions

Categories have a homotopical analogue in simplicial sets.

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A *quasi-category* is a simplicial set X with a (non-unique) lifting condition:

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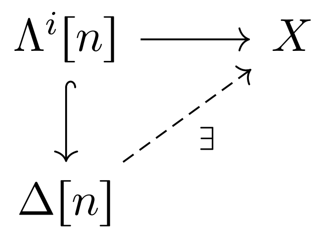

$0 < i < n$

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- Special Case: If all morphisms are invertible, we have a *Kan complex*—also defined by a non-unique lifting condition.

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Category	

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Category	Quasi-category

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Groupoid	
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  - ↳ fibrant objects: quasi-categories

Finding New Model Structure

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How do we find new model structures?

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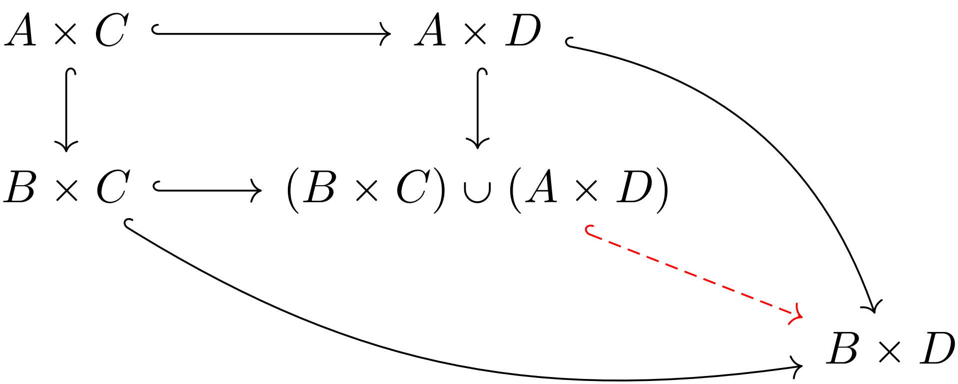
Cisinski gives us a way to find model structures with these properties.

Pushout-Product

Given $A \hookrightarrow B$ and $C \hookrightarrow D$, the induced map



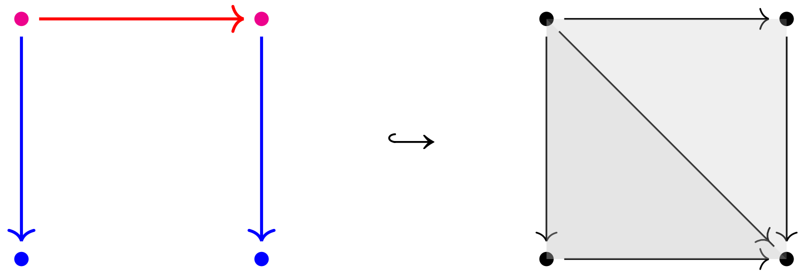
is their *pushout-product*, denoted $(A \hookrightarrow B) \,\square\, (C \hookrightarrow D)$.

Example

The pushout product of $0 \hookrightarrow \Delta[1]$ and $\partial\Delta[1] \hookrightarrow \Delta[1]$ is

$$(\Delta[1] \times \partial\Delta[1]) \cup (0 \times \Delta[1]) \longhookrightarrow \Delta[1] \times \Delta[1]$$

which looks like



Bdry $:= \{\partial\Delta[n] \hookrightarrow \Delta[n]\}_{n\geqslant 0}$ $\quad J = N(\bullet \leftrightarrows \bullet) \quad \partial J = 0 \cup 1$

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S a set of monomorphisms

$$
\begin{aligned}
A_J(S) := \quad & \text{Bdry} \,\square (0 \hookrightarrow J) \\
& \cup S \\
& \cup S \,\square (\partial J \hookrightarrow J) \\
& \cup (S \,\square (\partial J \hookrightarrow J)) \,\square (\partial J \hookrightarrow J) \\
& \vdots
\end{aligned}
$$

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Theorem (Cisinski)

- *For any set S of monomorphisms, there is a model structure whose fibrant objects are those with lifts against $A_J(S)$.*

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- *When the fibrant objects of a given model structure all lift against S, they also lift against $A_J(S)$.*

Example

If S is the set of spine extensions, we get the Joyal model structure.

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Idea

If we didn't know what a quasi-category was, we could let S be the spine extensions, and Cisinski's theory would tell us what a quasi-category should be.

Are We Done?

Now let $S = \{\mathcal{T} \hookrightarrow \Delta[n]\}$, the "two dimensional spine extensions."

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Analogy

Group presentations often don't tell us that much about a group.

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Yes, but...

Analogy

Group presentations often don't tell us that much about a group.

Similarly, even with the description from Cisinski's theory, there is still a lot we don't know about our model structure.

$$\text{Bdry} := \{\partial\Delta[n] \hookrightarrow \Delta[n]\}_{n \geqslant 0} \qquad J = N(\bullet \leftrightarrows \bullet)$$

$$A_J(\varnothing) := \text{Bdry} \,\square\, (0 \hookrightarrow J)$$

Theorem (Cisinski)

There is a model structure whose fibrant objects are those with lifts against $A_J(\varnothing)$; the minimal model structure *on* $\mathrm{Set}^{\Delta^{op}}$.

Minimal Model Structure

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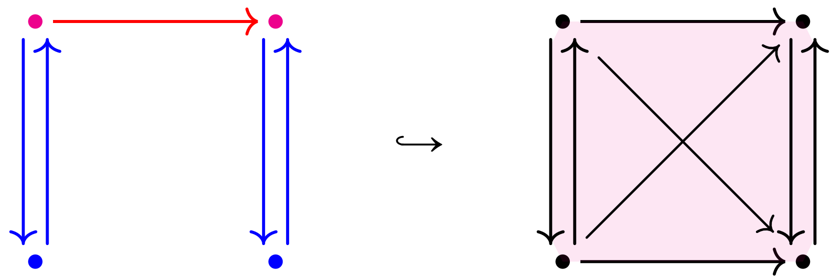
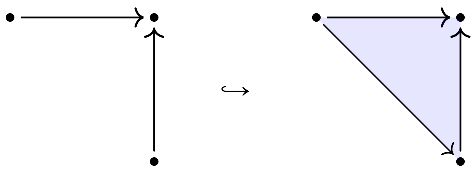
Ex: $n = 1$



Horns

- Simplex $=$
  $N(\bullet \to \bullet \to \ldots \to \bullet \to \bullet)$
- Face $=$ delete one vertex
- Horn $=$ union of all faces but one

$n = 2$ Horn Extension



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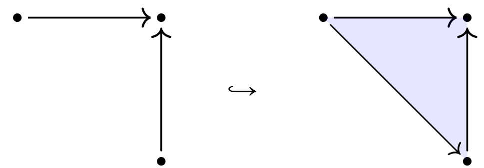
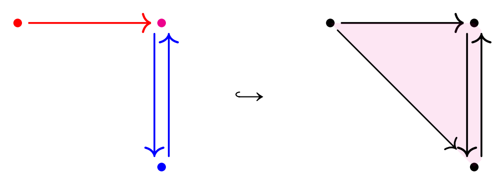


Iso-Horns

- Isoplex =
  $N(\bullet \to \ldots \to \bullet \leftrightarrows \bullet \quad \ldots \to \bullet)$
- Face = delete one vertex
- Horn = union of all faces but one, the one opposite a vertex of the isomorphism

$n = 2$ Iso-Horn Ext'n



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  - ↳ Codomains are categories, so n-simplices are equivalent to paths of arrows.

Interpretation

In category theory:

$$c \cong d \quad \Longrightarrow \quad \mathrm{Hom}(c, x) \cong \mathrm{Hom}(d, x) \text{ for all } x$$

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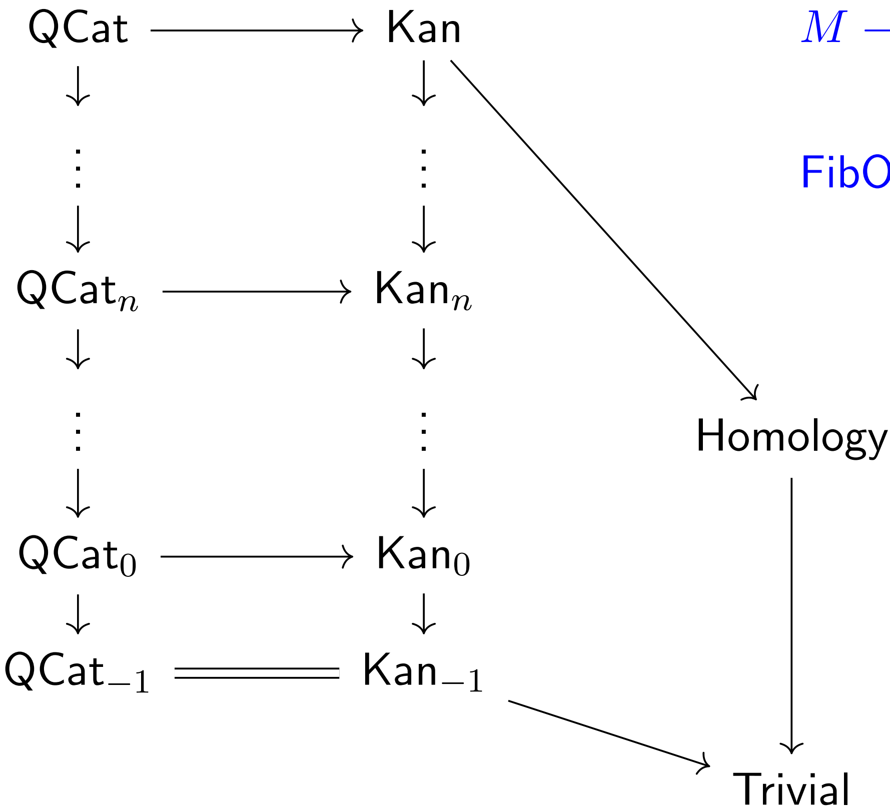
Similar thing happens: if two 0-simplices are “isomorphic,” then there is a correspondence between the n simplices of which they are vertices.

$\hookrightarrow$ $x \cong y \Longrightarrow x$ and y interact with the rest of X equivalently.

Model Structures on $\mathrm{Set}^{\Delta^{op}}$



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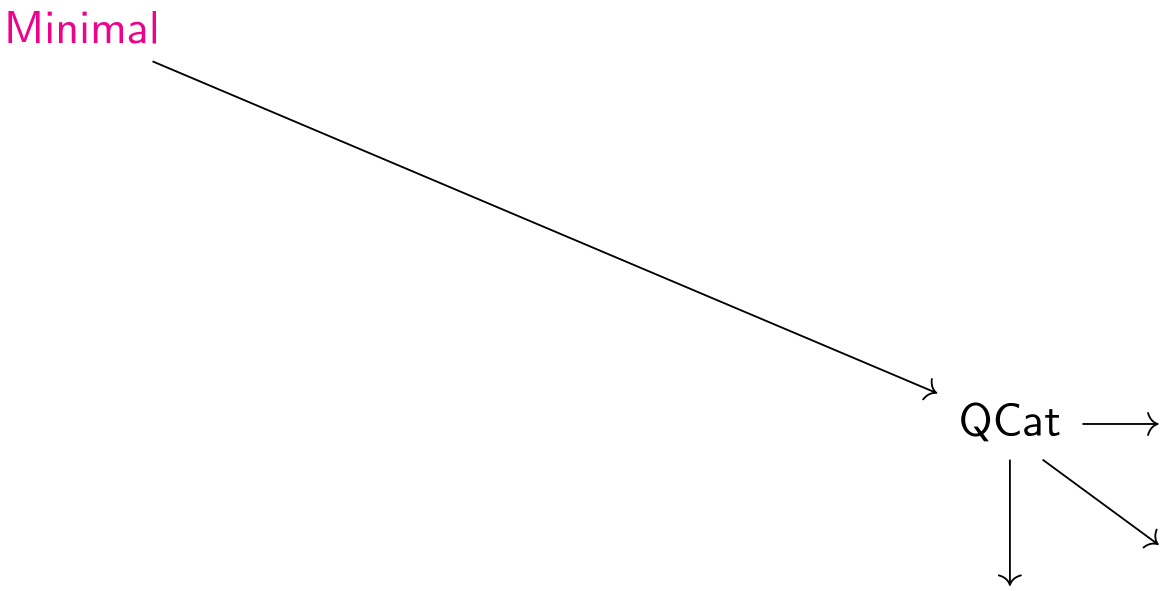

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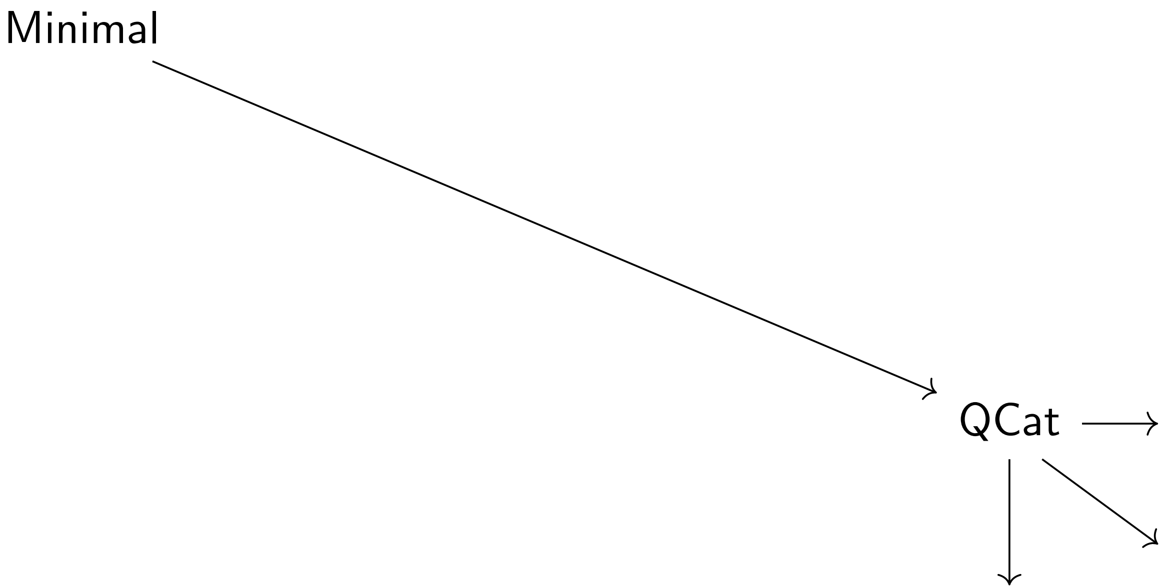

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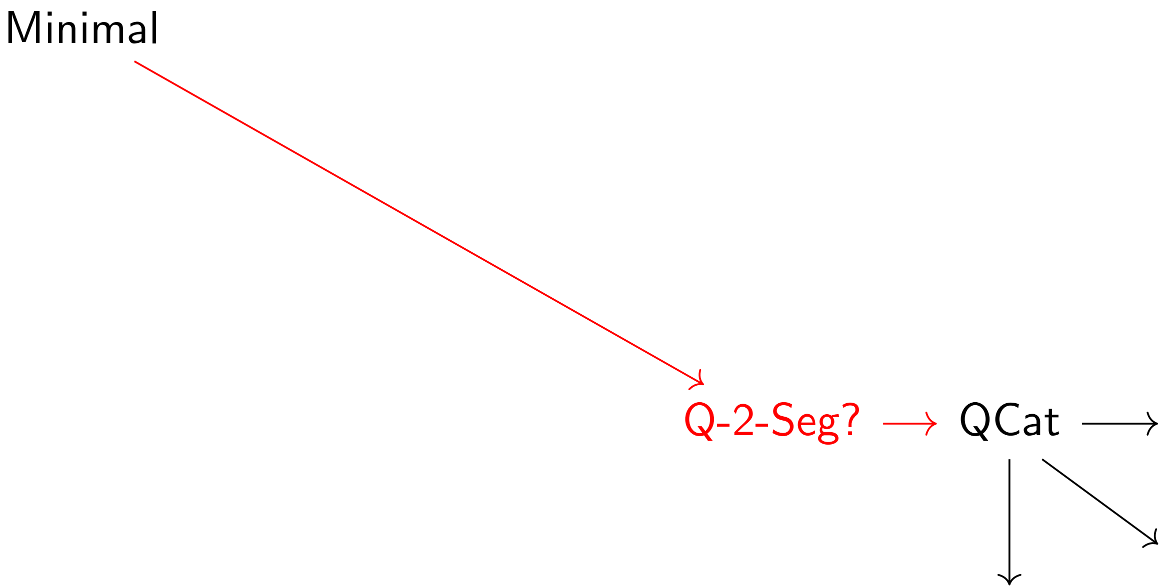

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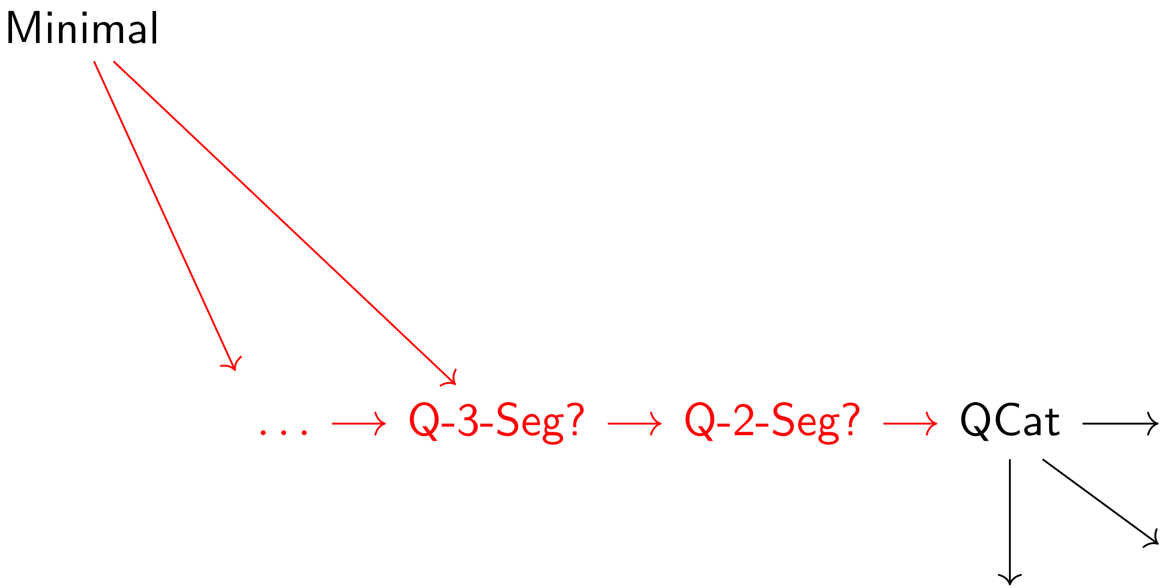

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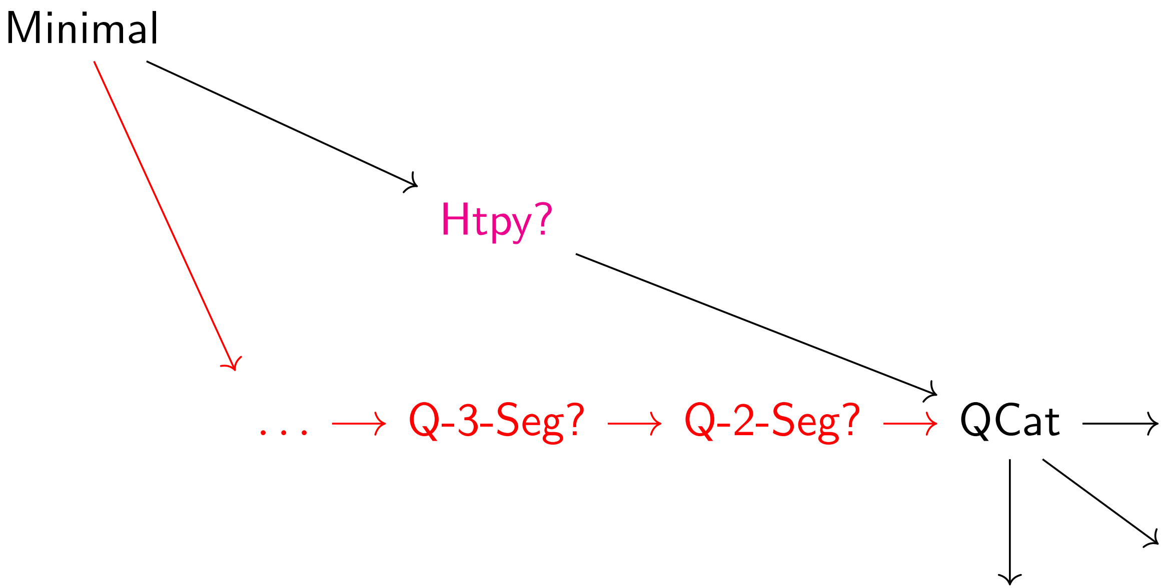

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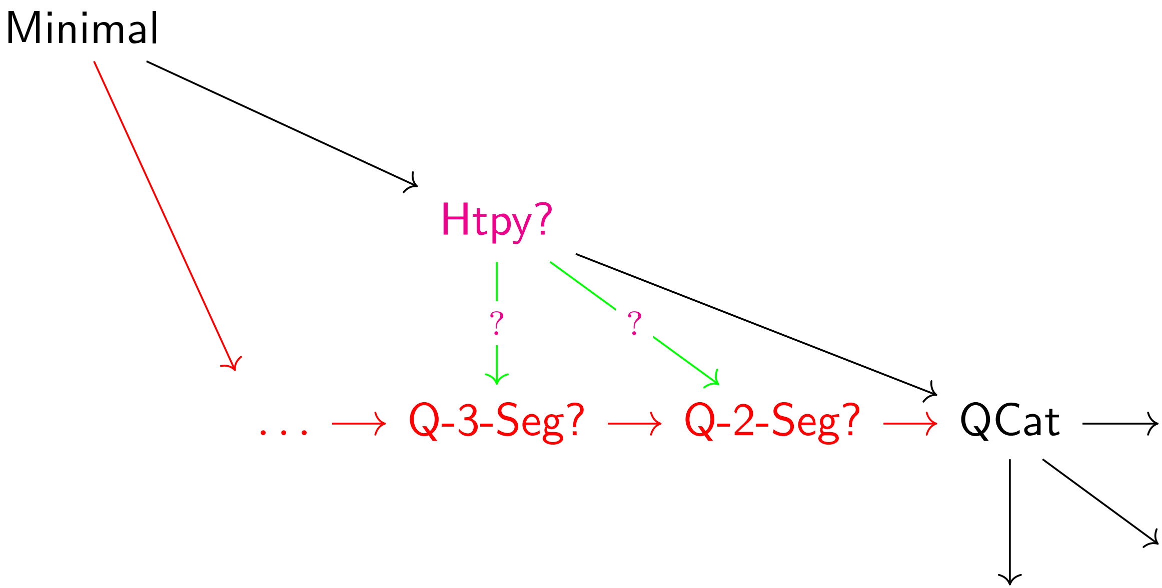

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