Coalgebras for enriched Hausdorff (and Vietoris) functors

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July 12, 2019

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Introduction

A QUICK REMINDER

Definition

For a functor $\mathsf{F}\colon \mathbf{C} \to \mathbf{C}$, one defines coalgebra

$$
\begin{tikzcd}
\mathsf{F}X & & \mathsf{F}Y \\
X \arrow[u, "c"] & & Y \arrow[u, "d"']
\end{tikzcd}
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For a functor $\mathsf{F}\colon \mathbf{C} \to \mathbf{C}$, one defines coalgebra homomorphism:

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The corresponding category we denote as $\mathsf{CoAlg}(\mathsf{F})$.

Theorem

The forgetful functor $\mathsf{CoAlg}(\mathsf{F}) \to \mathbf{C}$ *creates all colimits and those limits which are preserved by F.*

Recall

- The final coalgebra for $\mathsf{F}\colon \mathbf{C} \to \mathbf{C}$ is a fix-point of F. [a]

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[a] Joachim Lambek. “A fixpoint theorem for complete categories”. In: *Mathematische Zeitschrift* **103**.(2) (1968), pp. 151–161.

Recall

- The final coalgebra for $F\colon \mathbf{C} \to \mathbf{C}$ is a fix-point of F.
- The power-set functor $P\colon \mathbf{Set} \to \mathbf{Set}$ does not have a fix-point; hence P does not admit a final coalgebra. [a]

[a] Georg Cantor. “Über eine elementare Frage der Mannigfaltigkeitslehre”. In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* **1** (1891), pp. 75–78.

Motivation

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- The final coalgebra for $\mathsf{F}\colon \mathbf{C} \to \mathbf{C}$ is a fix-point of F.
- The power-set functor $\mathsf{P}\colon \mathbf{Set} \to \mathbf{Set}$ does not have a fix-point; hence P does not admit a final coalgebra.
- The finite power-set functor $\mathsf{P}_{\mathrm{fin}}\colon \mathbf{Set} \to \mathbf{Set}$ admits a final coalgebra (for instance, because $\mathsf{P}_{\mathrm{fin}}$ is finitary). [a]

[a] Michael Barr. "Terminal coalgebras in well-founded set theory". In: *Theoretical Computer Science* **114**.(2) (1993), pp. 299–315.

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- *Somehow more general*: the Vietoris functor $\mathsf{V}\colon \mathbf{CompHaus} \to \mathbf{CompHaus}$ admits a final coalgebra[a]

[a] Here: V preserves codirected limits. This result appears as an exercise in Ryszard Engelking. *General topology*. 2nd ed. Vol. 6. Sigma Series in Pure Mathematics. Berlin: Heldermann Verlag, 1989. viii + 529.

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- *Somehow more general*: the Vietoris functor $\mathsf{V} \colon \mathbf{CompHaus} \to \mathbf{CompHaus}$ admits a final coalgebra (and the same is true for $\mathsf{V} \colon \mathbf{PosComp} \to \mathbf{PosComp}$).[a][b]

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[a] Leopoldo Nachbin. *Topologia e Ordem*. University of Chicago Press, 1950.

[b] Dirk Hofmann, Renato Neves, and Pedro Nora. "Limits in categories of Vietoris coalgebras". In: *Mathematical Structures in Computer Science* **29**.(4) (2019), pp. 552–587.

Motivation

Recall

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- *Somehow more general*: the Vietoris functor $\mathsf{V}\colon \mathbf{CompHaus} \to \mathbf{CompHaus}$ admits a final coalgebra (and the same is true for $\mathsf{V}\colon \mathbf{PosComp} \to \mathbf{PosComp}$).
- *A bit more general*: the compact Vietoris functor $\mathsf{V}_{\mathrm{c}}\colon \mathbf{Top} \to \mathbf{Top}$ admits a final coalgebra.

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- The finite power-set functor $\mathsf{P}_{\mathrm{fin}}\colon \mathbf{Set} \to \mathbf{Set}$ admits a final coalgebra (for instance, because $\mathsf{P}_{\mathrm{fin}}$ is finitary).
- *Somehow more general*: the Vietoris functor $\mathsf{V}\colon \mathbf{CompHaus} \to \mathbf{CompHaus}$ admits a final coalgebra (and the same is true for $\mathsf{V}\colon \mathbf{PosComp} \to \mathbf{PosComp}$).
- *A bit more general*: the compact Vietoris functor $\mathsf{V}_{\mathrm{c}}\colon \mathbf{Top} \to \mathbf{Top}$ admits a final coalgebra.
- *A bit surprising(?)*: Also the lower Vietoris functor $\mathsf{V}\colon \mathbf{Top} \to \mathbf{Top}$ admits a final coalgebra.

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- the upset functor $\mathrm{Up}\colon \mathbf{Ord} \to \mathbf{Ord}$?
- liftings of **Set**-functors to **Met** (or, more general, to $\mathcal{V}$-**Cat**)? [a][b]

$$\begin{tikzcd} \mathcal{V}\text{-}\mathbf{Cat} \arrow[r, "\overline{T}"] \arrow[d] & \mathcal{V}\text{-}\mathbf{Cat} \arrow[d] \\ \mathbf{Set} \arrow[r, "T"'] & \mathbf{Set}. \end{tikzcd}$$

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[a] Paolo Baldan, Filippo Bonchi, Henning Kerstan, and Barbara König. "Coalgebraic Behavioral Metrics". In: *Logical Methods in Computer Science* **14**.(3) (2018), pp. 1860–5974.

[b] Adriana Balan, Alexander Kurz, and Jiří Velebil. "Extending set functors to generalised metric spaces". In: *Logical Methods in Computer Science* **15**.(1) (2019).

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What about “power functors” on other (topological) base categories? For instance,

- the upset functor $\mathrm{Up}\colon \mathbf{Ord} \to \mathbf{Ord}$?
- liftings of **Set**-functors to **Met** (or, more general, to $\mathcal{V}$-**Cat**)?
- (in particular) the Hausdorff functor? [a][b][c]

$$\mathrm{H}\colon \mathbf{Met} \longrightarrow \mathbf{Met}$$

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[a] Felix Hausdorff. *Grundzüge der Mengenlehre.* Leipzig: Veit & Comp, 1914. viii + 476.

[b] Dimitrie Pompeiu. “Sur la continuité des fonctions de variables complexes”. In: *Annales de la Faculté des Sciences de l’Université de Toulouse pour les Sciences Mathématiques et les Sciences Physiques. 2ième Série* **7**.(3) (1905), pp. 265–315.

[c] T. Birsan and Dan Tiba. “One hundred years since the introduction of the set distance by Dimitrie Pompeiu”. In: *System Modeling and Optimization*. Ed. by F. Pandolfi et al. Springer, 2006, pp. 35–39.

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What about "power functors" on other (topological) base categories? For instance,

- the upset functor $\mathrm{Up}\colon \mathbf{Ord} \to \mathbf{Ord}$?
- liftings of **Set**-functors to **Met** (or, more general, to $\mathcal{V}$-**Cat**)?
- (in particular) the Hausdorff functor? [a] [b]

$$\mathrm{H}\colon \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{V}\text{-}\mathbf{Cat}$$

Here $\mathrm{H}a(A,B) = \bigwedge_{y\in B} \bigvee_{x\in A} a(x,y)$, for a $\mathcal{V}$-category (X,a).

[a] Andrei Akhvlediani, Maria Manuel Clementino, and Walter Tholen. "On the categorical meaning of Hausdorff and Gromov distances, I". In: *Topology and its Applications* **157**.(8) (2010), pp. 1275–1295.

[b] Isar Stubbe. ""Hausdorff distance" via conical cocompletion". In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **51**.(1) (2010), pp. 51–76.

Some "powerful functors"

About the upset funtor

Theorem

Let X be a partially ordered set. Then there is no embedding $\varphi\colon \mathrm{Up}(X) \to X$. [a][b]

[a] Robert P. Dilworth and Andrew M. Gleason. "A generalized Cantor theorem". In: *Proceedings of the American Mathematical Society* **13**.(5) (1962), pp. 704–705.

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Corollary

The upset functor $\mathrm{Up}\colon \mathbf{Ord} \to \mathbf{Ord}$ *does not admit a final coalgebra.*

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Corollary

The upset functor $\mathsf{Up}\colon \mathbf{Ord} \to \mathbf{Ord}$ *does not admit a final coalgebra.*

Remark

The category $\mathsf{CoAlg}(\mathsf{Up})$ has equalisers.

Theorem

Consider the following commutative diagram of functors.

$$\begin{tikzcd} \mathbf{X} \arrow[r, "\overline{F}"] \arrow[d, "U"'] & \mathbf{X} \arrow[d, "U"] \\ \mathbf{A} \arrow[r, "F"'] & \mathbf{A} \end{tikzcd}$$

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2. *If* $\mathsf{U}\colon \mathbf{X} \to \mathbf{A}$ *is topological, then so is* $\mathsf{U}\colon \mathsf{CoAlg}(\overline{\mathsf{F}}) \to \mathsf{CoAlg}(\mathsf{F})$.

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   In particular, the category $\mathsf{CoAlg}(\overline{\mathsf{F}})$ *has limits of shape I if and only if* $\mathsf{CoAlg}(\mathsf{F})$ *has limits of shape I.*

Definition

Let $f\colon (X,a) \to (Y,b)$ be a $\mathcal{V}$-functor.

1. For every $A \subseteq X$, put $\uparrow^a A = \{y \in X \mid k \leq \bigvee_{x \in A} a(x,y)\}$.

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3. We consider the $\mathcal{V}$-category $\mathrm{H}X = \{A \subseteq X \mid A \text{ is increasing}\}$, equipped with $\mathrm{H}a(A,B) = \displaystyle\bigwedge_{y \in B} \bigvee_{x \in A} a(x,y)$, for all $A, B \in \mathrm{H}X$.

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4. The map $\mathrm{H}f\colon \mathrm{H}(X,a) \longrightarrow \mathrm{H}(Y,b)$ sends an increasing subset $A \subseteq X$ to $\uparrow^b f(A)$.
5. The functor H is part of a Kock–Zöberlein monad $\mathbb{H} = (\mathrm{H}, w, h)$ on $\mathcal{V}$-**Cat**.

$$h_X\colon X \longrightarrow \mathrm{H}X, \qquad w_X\colon \mathrm{HH}X \longrightarrow \mathrm{H}X.$$
$$x \longmapsto \uparrow x \qquad \mathcal{A} \longmapsto \bigcup \mathcal{A}$$

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6. $\mathbb{H}=(\mathrm{H},w,h)$ is a submonad of the covariant presheaf monad on $\mathcal{V}$-**Cat**; in fact, $\mathbb{H}$ is the monad of “conical limit weights”.

For metric spaces

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3. Furthermore, the Hausdorff functor preserves Cauchy completeness.
4. …

Ernest Michael. "Topologies on spaces of subsets". In: *Transactions of the American Mathematical Society* **71**.(1) (1951), pp. 152–182.

Sandro Levi, Roberto Lucchetti, and Jan Pelant. "On the infimum of the Hausdorff and Vietoris topologies". In: *Proceedings of the American Mathematical Society* **118**.(3) (1993), pp. 971–978.

Theorem

Let $\mathcal{V}$ be a non-trivial quantale and (X, a) be a $\mathcal{V}$-category. There is no embedding of type $\mathsf{H}(X, a) \to (X, a)$.

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Corollary

Let $\mathcal{V}$ be a non-trivial quantale. The Hausdorff functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$ does not admit a terminal coalgebra, neither does any possible restriction to a full subcategory of $\mathcal{V}$-$\mathbf{Cat}$.

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Remark

In particular, the (non-symmetric) Hausdorff functor on **Met** does not admit a terminal coalgebra, and the same applies to its restriction to the full subcategory of compact metric spaces.

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Remark

In particular, the (non-symmetric) Hausdorff functor on **Met** does not admit a terminal coalgebra, and the same applies to its restriction to the full subcategory of compact metric spaces. Passing to the symmetric version of the Hausdorff functor does not remedy the situation.

Adding topology

Extending the Ultrafilter monad

We assume that $\mathcal{V}$ is a completely distributive quantale, then

$$\xi\colon \mathsf{U}\mathcal{V} \longrightarrow \mathcal{V}, \quad \mathfrak{v} \longmapsto \bigwedge_{A\in\mathfrak{v}} \bigvee A$$

is the structure of an $\mathbb{U}$-algebra on $\mathcal{V}$ (the Lawson topology).

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Here:

$$\mathsf{U}a(\mathfrak{x},\mathfrak{y}) = \bigwedge_{A,B} \bigvee_{x,y} a(x,y), \qquad (X,a) \longmapsto (\mathsf{U}X, \mathsf{U}a).$$

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Its algebras are $\mathcal{V}$-categories equipped with a *compatible* compact Hausdorff topology [a][b]; we call them $\mathcal{V}$-categorical compact Hausdorff spaces, and denote the corresponding Eilenberg–Moore category by $\mathcal{V}$-**CatCH**.

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[a] Leopoldo Nachbin. *Topologia e Ordem*. University of Chicago Press, 1950.

[b] Walter Tholen. "Ordered topological structures". In: *Topology and its Applications* **156**.(12) (2009), pp. 2148–2157.

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Theorem

For an ordered set $(X, \leq)$ and a $\mathbb{U}$-algebra (X, α), the following are equivalent.

(i) $\alpha\colon (\mathsf{U}X, \mathsf{U}{\leq}) \to (X, \leq)$ *is monotone.*

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(i) $\alpha\colon (\mathsf{U}X,\mathsf{U}{\leq}) \to (X,\leq)$ *is monotone.*

(ii) $G_{\leq} \subseteq X \times X$ *is closed.*

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Its algebras are $\mathcal{V}$-categories equipped with a *compatible* compact Hausdorff topology; we call them $\mathcal{V}$-categorical compact Hausdorff spaces, and denote the corresponding Eilenberg–Moore category by $\mathcal{V}$-**CatCH**.

Theorem

For a $\mathcal{V}$-category (X, a) and a $\mathbb{U}$-algebra (X, α), the following are equivalent.

(i) $\alpha\colon \mathsf{U}(X, a) \to (X, a)$ *is a $\mathcal{V}$-functor.*

(ii) $a\colon (X, \alpha) \times (X, \alpha) \to (\mathcal{V}, \xi_{\leq})$ *is continuous.*

Open “(”

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Theorem

For an ordered compact Hausdorff space X, the ordered set X is directed complete.

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Proof.

$$\begin{tikzcd} \mathbf{OrdCH} \arrow[r, "\mathsf{K}"] \arrow[dr, "|-|"'] & \mathbf{Top} \arrow[d, "\mathsf{S}"] \\ & \mathbf{Ord} \end{tikzcd}$$

$$\begin{gathered} \mathsf{K}(\mathsf{X}) \text{ is sober}, \dots \\ \Downarrow \\ |X| \text{ is directed complete} \end{gathered}$$

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For a metric compact Hausdorff space X, the metric space X is Cauchy complete.

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Theorem

For a metric *compact Hausdorff space X, the metric space X is* Cauchy *complete.*

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$$\begin{tikzcd} \mathbf{MetCH} \arrow[r, "\mathsf{K}"] \arrow[dr, "|-|"'] & \mathbf{App} \arrow[d, "\mathsf{S}"] \\ & \mathbf{Met} \end{tikzcd} \qquad \begin{array}{c} \mathsf{K}(X) \text{ is sober, } \dots \\ \Downarrow \\ |X| \text{ is directed complete} \end{array}$$

$\square$

Approach space = “metric” topological space.

Robert Lowen. “Approach spaces: a common supercategory of TOP and MET”. In: *Mathematische Nachrichten* **141**.(1) (1989), pp. 183–226.

Bernhard Banaschewski, Robert Lowen, and Cristophe Van Olmen. “Sober approach spaces”. In: *Topology and its Applications* **153**.(16) (2006), pp. 3059–3070.

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Theorem (Lawvere (1973))

A metric space X is Cauchy-complete if and only if every left adjoint distributors $\varphi\colon 1 \longrightarrow\!\!\!\!\!\!\!\circ\;\; X$ is representable (i.e. $\varphi = x_*$).

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Example

Every discrete metric space is Cauchy complete (any compact Hausdorff topology).

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For a metric compact Hausdorff space X, the metric space X is Cauchy complete.

Proof.

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Corollary

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Example

(UX, Ud) is Cauchy complete. Consider $(UX, Ud, m_X) \dots$ and close ")"

Lemma

Let (X, a, α) be a $\mathcal{V}$-categorical compact Hausdorff space and $A, B \subseteq X$ so that $A \cap B = \varnothing$, A is increasing and compact in $(X, \alpha_{\leq})^{\mathrm{op}}$ and B is compact in $(X, \alpha_{\leq})$. Then there exists some $u \ll k$ so that, for all $x \in A$ and $y \in B$, $u \nleq a(x, y)$.

Towards "Urysohn"

Lemma

Let (X, a, α) be a $\mathcal{V}$-categorical compact Hausdorff space and $A, B \subseteq X$ so that $A \cap B = \varnothing$, A is increasing and compact in $(X, \alpha_{\leq})^{\mathrm{op}}$ and B is compact in $(X, \alpha_{\leq})$. Then there exists some $u \ll k$ so that, for all $x \in A$ and $y \in B$, $u \nleq a(x, y)$.

Corollary

For every compact subset $A \subseteq X$ of $(X, \alpha_{\leq})^{\mathrm{op}}$, $\uparrow^{a} A = \uparrow^{\leq} A$. In particular, for every closed subset $A \subseteq X$ of (X, α), $\uparrow^{a} A = \uparrow^{\leq} A$.

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Theorem (Nachbin)

Let $A \subseteq X$ be closed and decreasing and $B \subseteq X$ be closed and increasing with $A \cap B = \varnothing$. Then there exist $V \subseteq X$ open and co-increasing and $W \subseteq X$ open and co-decreasing with

$$A \subseteq V, \quad B \subseteq W, \quad V \cap W = \varnothing.$$

Definition

For a $\mathcal{V}$-categorical compact Hausdorff space $X = (X, a, \alpha)$, we put

$$\mathrm{H}X = \{A \subseteq X \mid A \text{ is closed and increasing}\}$$

with the restriction of the Hausdorff structure to $\mathrm{H}X$ and the hit-and-miss topology (Vietoris topology). That is, the topology generated by the sets

$$V^{\Diamond} = \{A \in \mathrm{H}X \mid A \cap V \neq \varnothing\} \qquad (V \text{ open, co-increasing})$$

and

$$W^{\Box} = \{A \in \mathrm{H}X \mid A \subseteq W\} \qquad (W \text{ open, co-decreasing}).$$

The Hausdorff monad (again)

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Proposition

For every $\mathcal{V}$-categorical compact Hausdorff space X, $\mathrm{H}X$ is a $\mathcal{V}$-categorical compact Hausdorff space.

Compare with:

For a compact metric space, the Hausdorff metric induces the Vietoris topology.

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The construction above defines a functor $\mathrm{H}\colon \mathcal{V}\text{-}\mathbf{CatCH} \longrightarrow \mathcal{V}\text{-}\mathbf{CatCH}$.

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In fact, we obtain a Kock–Zöberlein monad.

Theorem (Lawvere (1973))

A metric space X is Cauchy-complete if and only if every left adjoint distributor $\varphi\colon 1 \longrightarrow\!\!\!\!\!\!\!\circ\;\; X$ is representable (i.e. $\varphi = x_*$).

Compact $\mathcal{V}$-categories

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Definition

For a $\mathcal{V}$-category X, $A \subseteq X$ and $x \in X$, we define $x \in \overline{A}$ whenever "x represents a left adjoint distributor $1 \longrightarrow\!\!\!\!\!\!\!\circ\;\; A$".

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- Under suitable conditions, this closure operator is topological.

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Remark

- Under suitable conditions, this closure operator is topological.
- Moreover, if X is separated, then this topology is Hausdorff.
- With $\mathcal{V}$-$\mathbf{Cat}_{\mathrm{ch}}$ denoting the full subcategory of $\mathcal{V}$-$\mathbf{Cat}_{\mathrm{sep}}$ defined by those $\mathcal{V}$-categories with compact topology, we obtain a functor $\mathcal{V}$-$\mathbf{Cat}_{\mathrm{ch}} \to \mathcal{V}$-$\mathbf{CatCH}$.

Proposition

For the $\mathcal{V}$-categorical compact Hausdorff space induced by a compact separated $\mathcal{V}$-category X, the hit-and-miss topology on $\mathrm{H}X$ coincides with the topology induced by the Hausdorff structure on $\mathrm{H}X$.

Proposition

For the $\mathcal{V}$-categorical compact Hausdorff space induced by a compact separated $\mathcal{V}$-category X, the hit-and-miss topology on $\mathsf{H}X$ *coincides with the topology induced by the Hausdorff structure on* $\mathsf{H}X$.

Theorem

The functor $\mathsf{H}\colon \mathcal{V}\text{-}\mathbf{Cat} \to \mathcal{V}\text{-}\mathbf{Cat}$ *restricts to the category* $\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{ch}}$*, moreover, the diagram*

$$\begin{tikzcd}
\mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{ch}} \arrow[r, "\mathsf{H}"] \arrow[d] & \mathcal{V}\text{-}\mathbf{Cat}_{\mathrm{ch}} \arrow[d] \\
\mathcal{V}\text{-}\mathbf{CatCH} \arrow[r, "\mathsf{H}"'] & \mathcal{V}\text{-}\mathbf{CatCH}
\end{tikzcd}$$

commutes.

Proposition

The diagrams of functors commutes.

$$\begin{tikzcd}
\mathbf{OrdCH} \arrow[r, "H"] \arrow[d, bend right] & \mathbf{OrdCH} \arrow[d, bend right] \\
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*The Hausdorff functor on $\mathcal{V}$-**CatCH** preserves codirected initial cones with respect to the forgetful functor $\mathcal{V}$-**CatCH** $\to$ **CompHaus**.*

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Theorem

The Hausdorff functor $\mathsf{H}\colon \mathcal{V}$-**CatCH** $\to \mathcal{V}$-**CatCH** *preserves codirected limits.*

Theorem

For $\mathsf{H}\colon \mathcal{V}\text{-}\mathbf{CatCH} \to \mathcal{V}\text{-}\mathbf{CatCH}$*, the forgetful functor* $\mathsf{CoAlg}(\mathsf{H}) \to \mathcal{V}\text{-}\mathbf{CatCH}$ *is comonadic. Moreover,* $\mathcal{V}\text{-}\mathbf{CatCH}$ *has equalisers and is therefore complete.*

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Theorem

The category of coalgebras of a Hausdorff polynomial functor on $\mathcal{V}\text{-}\mathbf{CatCH}$ *is (co)complete.*

Definition

We call a functor Hausdorff polynomial whenever it belongs to the smallest class of endofunctors on $\mathcal{V}\text{-}\mathbf{Cat}$ that contains the identity functor, all constant functors and is closed under composition with H, products and sums of functors.

Recall

An ordered compact Hausdorff space is a Priestley space whenever the cone $\mathbf{PosComp}(X, \mathbf{2})$ is an initial monocone. [a][b]

[a] Hilary A. Priestley. "Representation of distributive lattices by means of ordered stone spaces". In: *Bulletin of the London Mathematical Society* **2**.(2) (1970), pp. 186–190.

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Priestley spaces

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Assumption

From now on we assume that the maps $\hom(u, -): \mathcal{V} \to \mathcal{V}$ are continuous.

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Proposition

Let X be a $\mathcal{V}$-categorical compact Hausdorff space. Consider a $\mathcal{V}$-subcategory $\mathcal{R} \subseteq \mathcal{V}^X$ that is closed under finite weighted limits and such that $(\psi \colon X \to \mathcal{V}^{\mathrm{op}})_{\psi \in \mathcal{R}}$ is initial with respect to $\mathcal{V}\text{-}\mathbf{CatCH} \to \mathbf{CompHaus}$.

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Then the cone $(\psi^{\Diamond} \colon \mathsf{H}X \to \mathcal{V}^{\mathrm{op}})_{\psi \in \mathcal{R}}$ is initial with respect to $\mathcal{V}$-$\mathbf{CatCH} \to \mathbf{CompHaus}$.

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*Then the cone $(\psi^{\Diamond} \colon \mathrm{H}X \to \mathcal{V}^{\mathrm{op}})_{\psi \in \mathcal{R}}$ is initial with respect to $\mathcal{V}$-**CatCH** $\to$ **CompHaus**.*

Corollary

*The Hausdorff functor restricts to a functor $\mathrm{H} \colon \mathcal{V}$-**Priest** $\to \mathcal{V}$-**Priest**, hence the Hausdorff monad $\mathbb{H}$ restricts to $\mathcal{V}$-**Priest**.*

Duality theory

Theorem

$\mathrm{CoAlg(H)} \simeq \mathbf{DLO}^{\mathrm{op}}$ *(distributive lattices with operator).*[a][b]

[a] Alejandro Petrovich. “Distributive lattices with an operator”. In: *Studia Logica* **56**.(1-2) (1996), pp. 205–224.

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Theorem

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From 2 to $[0,1]$

Theorem

$C\colon \mathbf{StablyComp}_{\mathbb{V}} \longrightarrow \mathrm{LaxMon}([0,1]\text{-}\mathbf{FinSup})^{\mathrm{op}}$ *is fully faithful.*[a]

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Remark

- $A \subseteq X$ closed $(1 \nrightarrow X)$ $\quad\leftrightsquigarrow\quad$ $\Phi : CX \longrightarrow [0,1]$.

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- $\varphi\colon X \longrightarrow [0,1]$ $\quad (1 \overset{\varphi}{\longrightarrow\!\!\!\!\!\circ\;} X)$ $\quad\leftrightsquigarrow\quad$ $\Phi : CX \longrightarrow [0,1]$.
- $\varphi\colon X \longrightarrow [0,1]$ is irreducible(?) $\quad\iff\quad$ Φ is ????

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[a] Maria Manuel Clementino and Dirk Hofmann. “Lawvere completeness in topology”. In: *Applied Categorical Structures* **17**.(2) (2009), pp. 175–210.

[b] Dirk Hofmann and Isar Stubbe. “Towards Stone duality for topological theories”. In: *Topology and its Applications* **158**.(7) (2011), pp. 913–925.

“Irreducible” distributors

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For Łukasiewicz $\otimes = \odot$

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Furthermore, the diagram

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Conclusion: $\varphi\colon 1 \multimap X$ is left adjoint $\iff$ Φ preserves finite weighted limits.