

COALGEBRAS FOR ENRICHED HAUSDORFF (AND VIETORIS) FUNCTORS

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INTRODUCTION

A QUICK REMINDER

Definition

For a functor $F: \mathbf{C} \rightarrow \mathbf{C}$, one defines coalgebra

$$\begin{array}{ccc} FX & & FY \\ \uparrow c & & \uparrow d \\ X & & Y \end{array}$$

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Theorem

The forgetful functor $\text{CoAlg}(F) \rightarrow \mathbf{C}$ creates all colimits and those limits which are preserved by F .

Recall

- The final coalgebra for $F: \mathbf{C} \rightarrow \mathbf{C}$ is a fix-point of F .^a

^aJoachim Lambek. “A fixpoint theorem for complete categories”. In: *Mathematische Zeitschrift* **103**.(2) (1968), pp. 151–161.

Recall

- The final coalgebra for $F: \mathbf{C} \rightarrow \mathbf{C}$ is a fix-point of F .
- The power-set functor $P: \mathbf{Set} \rightarrow \mathbf{Set}$ does not have a fix-point; hence P does not admit a final coalgebra.^a

^aGeorg Cantor. “Über eine elementare Frage der Mannigfaltigkeitslehre”. In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* **1** (1891), pp. 75–78.

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- The final coalgebra for $F: \mathbf{C} \rightarrow \mathbf{C}$ is a fix-point of F .
- The power-set functor $P: \mathbf{Set} \rightarrow \mathbf{Set}$ does not have a fix-point; hence P does not admit a final coalgebra.
- The finite power-set functor $P_{\text{fin}}: \mathbf{Set} \rightarrow \mathbf{Set}$ admits a final coalgebra (for instance, because P_{fin} is finitary).^a

^aMichael Barr. “Terminal coalgebras in well-founded set theory”. In: *Theoretical Computer Science* 114.(2) (1993), pp. 299–315.

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- *Somehow more general*: the Vietoris functor $V: \mathbf{CompHaus} \rightarrow \mathbf{CompHaus}$ admits a final coalgebra^a

^aHere: V preserves codirected limits. This result appears as an exercise in Ryszard Engelking. *General topology*. 2nd ed. Vol. 6. Sigma Series in Pure Mathematics. Berlin: Heldermann Verlag, 1989. viii + 529.

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- *Somehow more general*: the Vietoris functor $V: \mathbf{CompHaus} \rightarrow \mathbf{CompHaus}$ admits a final coalgebra (and the same is true for $V: \mathbf{PosComp} \rightarrow \mathbf{PosComp}$).^{ab}

^aLeopoldo Nachbin. *Topologia e Ordem*. University of Chicago Press, 1950.

^bDirk Hofmann, Renato Neves, and Pedro Nora. “Limits in categories of Vietoris coalgebras”. In: *Mathematical Structures in Computer Science* 29.(4) (2019), pp. 552–587.

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- *A bit more general*: the compact Vietoris functor $V_c: \mathbf{Top} \rightarrow \mathbf{Top}$ admits a final coalgebra.

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- *Somehow more general*: the Vietoris functor $V: \mathbf{CompHaus} \rightarrow \mathbf{CompHaus}$ admits a final coalgebra (and the same is true for $V: \mathbf{PosComp} \rightarrow \mathbf{PosComp}$).
- *A bit more general*: the compact Vietoris functor $V_c: \mathbf{Top} \rightarrow \mathbf{Top}$ admits a final coalgebra.
- *A bit surprising(?)*: Also the lower Vietoris functor $V: \mathbf{Top} \rightarrow \mathbf{Top}$ admits a final coalgebra.

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- liftings of **Set**-functors to **Met** (or, more general, to $\mathcal{V}\text{-Cat}$)? ^{ab}

$$\begin{array}{ccc} \mathcal{V}\text{-Cat} & \xrightarrow{\bar{T}} & \mathcal{V}\text{-Cat} \\ \downarrow & & \downarrow \\ \mathbf{Set} & \xrightarrow{T} & \mathbf{Set}. \end{array}$$

^aPaolo Baldan, Filippo Bonchi, Henning Kerstan, and Barbara König. “Coalgebraic Behavioral Metrics”. In: *Logical Methods in Computer Science* **14**.(3) (2018), pp. 1860–5974.

^bAdriana Balan, Alexander Kurz, and Jiří Velebil. “Extending set functors to generalised metric spaces”. In: *Logical Methods in Computer Science* **15**.(1) (2019).

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- the upset functor $\text{Up}: \mathbf{Ord} \rightarrow \mathbf{Ord}$?
- liftings of **Set**-functors to **Met** (or, more general, to \mathcal{V} -**Cat**)?
- (in particular) the Hausdorff functor? ^{abc}

$$H: \mathbf{Met} \longrightarrow \mathbf{Met}$$

^aFelix Hausdorff. *Grundzüge der Mengenlehre*. Leipzig: Veit & Comp, 1914. viii + 476.

^bDimitrie Pompeiu. “Sur la continuité des fonctions de variables complexes”. In: *Annales de la Faculté des Sciences de l’Université de Toulouse pour les Sciences Mathématiques et les Sciences Physiques*. 2ième Série 7.(3) (1905), pp. 265–315.

^cT. Birsan and Dan Tiba. “One hundred years since the introduction of the set distance by Dimitrie Pompeiu”. In: *System Modeling and Optimization*. Ed. by F. Pandolfi et al. Springer, 2006, pp. 35–39.

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- (in particular) the Hausdorff functor? ^{ab}

$$H: \mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Cat}$$

Here $H_a(A, B) = \bigwedge_{y \in B} \bigvee_{x \in A} a(x, y)$, for a \mathcal{V} -category (X, a) .

^aAndrei Akhvediani, Maria Manuel Clementino, and Walter Tholen. “On the categorical meaning of Hausdorff and Gromov distances, I”. In: *Topology and its Applications* **157**.(8) (2010), pp. 1275–1295.

^bIsar Stubbe. ““Hausdorff distance” via conical cocompletion”. In: *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **51**.(1) (2010), pp. 51–76.

SOME "POWERFUL FUNCTORS"

Theorem

Let X be a partially ordered set. Then there is no embedding $\varphi: \text{Up}(X) \rightarrow X$.^{ab}

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Remark

The category $\text{CoAlg}(\text{Up})$ has equalisers.

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Consider the following commutative diagram of functors.

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\bar{F}} & \mathbf{X} \\ \downarrow U & & \downarrow U \\ \mathbf{A} & \xrightarrow{F} & \mathbf{A} \end{array}$$

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2. If $U: \mathbf{X} \rightarrow \mathbf{A}$ is topological, then so is $U: \text{CoAlg}(\bar{F}) \rightarrow \text{CoAlg}(F)$.

In particular, the category $\text{CoAlg}(\bar{F})$ has limits of shape I if and only if $\text{CoAlg}(F)$ has limits of shape I .

Definition

Let $f: (X, a) \rightarrow (Y, b)$ be a \mathcal{V} -functor.

1. For every $A \subseteq X$, put $\uparrow^a A = \{y \in X \mid k \leq \bigvee_{x \in A} a(x, y)\}$.

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3. We consider the \mathcal{V} -category $HX = \{A \subseteq X \mid A \text{ is increasing}\}$, equipped with
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4. The map $Hf: H(X, a) \rightarrow H(Y, b)$ sends an increasing subset $A \subseteq X$ to $\uparrow^b f(A)$.
5. The functor H is part of a Kock–Zöberlein monad $\mathbb{H} = (H, w, \hat{h})$ on \mathcal{V} -Cat.

$$\hat{h}_X: X \longrightarrow HX,$$

$$x \longmapsto \uparrow x$$

$$w_X: HHX \longrightarrow HX.$$

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5. The functor H is part of a Kock–Zöberlein monad $\mathbb{H} = (H, \omega, \eta)$ on \mathcal{V} -Cat.
6. $\mathbb{H} = (H, \omega, \eta)$ is a submonad of the covariant presheaf monad on \mathcal{V} -Cat; in fact, \mathbb{H} is the monad of “conical limit weights”.

SOME CLASSICAL RESULTS

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2. Hence, the Hausdorff functor sends compact metric spaces to compact metric spaces.
3. Furthermore, the Hausdorff functor preserves Cauchy completeness.
4. ...

Ernest Michael. "Topologies on spaces of subsets". In: *Transactions of the American Mathematical Society* **71**.(1) (1951), pp. 152–182.

Sandro Levi, Roberto Lucchetti, and Jan Pelant. "On the infimum of the Hausdorff and Vietoris topologies". In: *Proceedings of the American Mathematical Society* **118**.(3) (1993), pp. 971–978.

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Let \mathcal{V} be a non-trivial quantale and (X, a) be a \mathcal{V} -category. There is no embedding of type $H(X, a) \rightarrow (X, a)$.

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Let \mathcal{V} be a non-trivial quantale. The Hausdorff functor $H: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Cat}$ does not admit a terminal coalgebra, neither does any possible restriction to a full subcategory of $\mathcal{V}\text{-Cat}$.

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Remark

In particular, the (non-symmetric) Hausdorff functor on **Met** does not admit a terminal coalgebra, and the same applies to its restriction to the full subcategory of compact metric spaces.

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Remark

In particular, the (non-symmetric) Hausdorff functor on **Met** does not admit a terminal coalgebra, and the same applies to its restriction to the full subcategory of compact metric spaces. Passing to the symmetric version of the Hausdorff functor does not remedy the situation.

ADDING TOPOLOGY

Extending the Ultrafilter monad

We assume that \mathcal{V} is a completely distributive quantale, then

$$\xi: \mathbb{U}\mathcal{V} \longrightarrow \mathcal{V}, \quad \mathfrak{v} \longmapsto \bigwedge_{A \in \mathfrak{v}} \bigvee A$$

is the structure of an \mathbb{U} -algebra on \mathcal{V} (the Lawson topology).

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Here:

$$\mathbb{U}a(x, \eta) = \bigwedge_{A, B} \bigvee_{x, y} a(x, y), \quad (X, a) \longmapsto (\mathbb{U}X, \mathbb{U}a).$$

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Its algebras are \mathcal{V} -categories equipped with a *compatible* compact Hausdorff topology^{ab}; we call them **\mathcal{V} -categorical compact Hausdorff spaces**, and denote the corresponding Eilenberg–Moore category by $\mathcal{V}\text{-CatCH}$.

^aLeopoldo Nachbin. *Topologia e Ordem*. University of Chicago Press, 1950.

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Theorem

For an ordered set (X, \leq) and a \mathbb{U} -algebra (X, α) , the following are equivalent.

- (i) $\alpha: (\mathbb{U}X, \mathbb{U}\leq) \rightarrow (X, \leq)$ is monotone.

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- (ii) $G_{\leq} \subseteq X \times X$ is closed.

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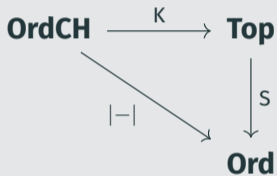
- (i) $\alpha: \mathbb{U}(X, a) \rightarrow (X, a)$ is a \mathcal{V} -functor.
- (ii) $a: (X, \alpha) \times (X, \alpha) \rightarrow (\mathcal{V}, \xi_{\leq})$ is continuous.

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Proof.

$K(X)$ is sober, ...

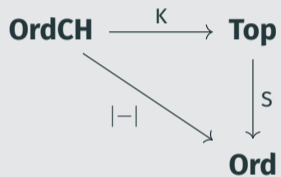


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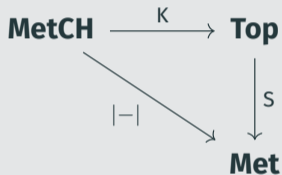


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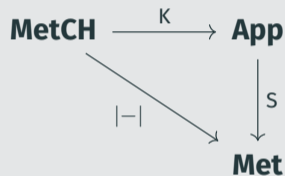


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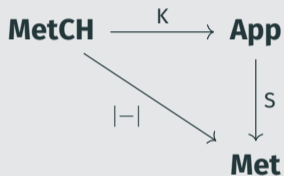
Approach space = “metric” topological space.

Robert Lowen. “Approach spaces: a common supercategory of TOP and MET”. In: *Mathematische Nachrichten* **141**.(1) (1989), pp. 183–226.

Bernhard Banaschewski, Robert Lowen, and Christophe Van Olmen. “Sober approach spaces”. In: *Topology and its Applications* **153**.(16) (2006), pp. 3059–3070.

Theorem

For a *metric* compact Hausdorff space X , the metric space X is *Cauchy* complete.

Proof.

$K(X)$ is sober, ...



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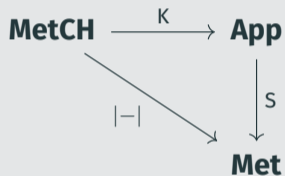
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A metric space X is Cauchy-complete if and only if every left adjoint distributor $\varphi: \mathbf{1} \multimap \rightarrow X$ is representable (i.e. $\varphi = x_*$).

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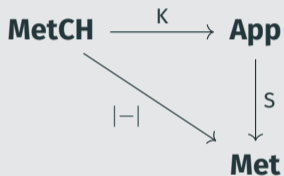
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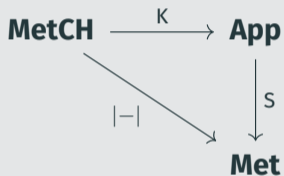
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**Corollary**

Every *compact metric* space is *Cauchy* complete.

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Corollary

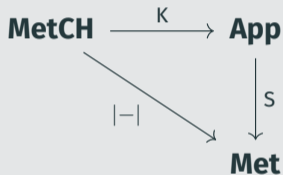
Every *compact metric space* is *Cauchy complete*.

Example

Every discrete metric space is *Cauchy complete* (any compact Hausdorff topology).

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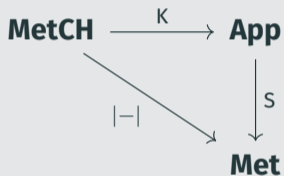
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(UX, U_d) is *Cauchy* complete.

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(UX, Ud) is *Cauchy* complete. Consider $(UX, Ud, m_X) \dots$ and close “)”

Lemma

Let (X, a, α) be a \mathcal{V} -categorical compact Hausdorff space and $A, B \subseteq X$ so that $A \cap B = \emptyset$, A is increasing and compact in $(X, \alpha_{\leq})^{\text{op}}$ and B is compact in (X, α_{\leq}) . Then there exists some $u \ll k$ so that, for all $x \in A$ and $y \in B$, $u \not\leq a(x, y)$.

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Corollary

For every compact subset $A \subseteq X$ of $(X, \alpha_{\leq})^{\text{op}}$, $\uparrow^a A = \uparrow^{\leq} A$. In particular, for every closed subset $A \subseteq X$ of (X, α) , $\uparrow^a A = \uparrow^{\leq} A$.

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Theorem (Nachbin)

Let $A \subseteq X$ be closed and decreasing and $B \subseteq X$ be closed and increasing with $A \cap B = \emptyset$. Then there exist $V \subseteq X$ open and co-increasing and $W \subseteq X$ open and co-decreasing with

$$A \subseteq V, \quad B \subseteq W, \quad V \cap W = \emptyset.$$

THE HAUSDORFF MONAD (AGAIN)

Definition

For a \mathcal{V} -categorical compact Hausdorff space $X = (X, a, \alpha)$, we put

$$HX = \{A \subseteq X \mid A \text{ is closed and increasing}\}$$

with the restriction of the Hausdorff structure to HX and the **hit-and-miss topology** (Vietoris topology). That is, the topology generated by the sets

$$V^\diamond = \{A \in HX \mid A \cap V \neq \emptyset\} \quad (V \text{ open, co-increasing})$$

and

$$W^\square = \{A \in HX \mid A \subseteq W\} \quad (W \text{ open, co-decreasing}).$$

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Proposition

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Compare with:

For a compact metric space, the Hausdorff metric induces the Vietoris topology.

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In fact, we obtain a Kock-Zöberlein monad.

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- Under suitable conditions, this closure operator is topological.
- Moreover, if X is separated, then this topology is Hausdorff.
- With $\mathcal{V}\text{-Cat}_{\text{ch}}$ denoting the full subcategory of $\mathcal{V}\text{-Cat}_{\text{sep}}$ defined by those \mathcal{V} -categories with compact topology, we obtain a functor $\mathcal{V}\text{-Cat}_{\text{ch}} \rightarrow \mathcal{V}\text{-CatCH}$.

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Theorem

The functor $H: \mathcal{V}\text{-Cat} \rightarrow \mathcal{V}\text{-Cat}$ restricts to the category $\mathcal{V}\text{-Cat}_{\text{ch}}$, moreover, the diagram

$$\begin{array}{ccc} \mathcal{V}\text{-Cat}_{\text{ch}} & \xrightarrow{H} & \mathcal{V}\text{-Cat}_{\text{ch}} \\ \downarrow & & \downarrow \\ \mathcal{V}\text{-CatCH} & \xrightarrow{H} & \mathcal{V}\text{-CatCH} \end{array}$$

commutes.

Proposition

The diagrams of functors commutes.

$$\begin{array}{ccc} \mathbf{OrdCH} & \xrightarrow{H} & \mathbf{OrdCH} \\ \left(\begin{array}{c} \downarrow \\ \uparrow \end{array} \right) & & \left(\begin{array}{c} \downarrow \\ \uparrow \end{array} \right) \\ \mathcal{V}\text{-CatCH} & \xrightarrow{H} & \mathcal{V}\text{-CatCH} \end{array}$$

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For $H: \mathcal{V}\text{-CatCH} \rightarrow \mathcal{V}\text{-CatCH}$, the forgetful functor $\text{CoAlg}(H) \rightarrow \mathcal{V}\text{-CatCH}$ is comonadic. Moreover, $\mathcal{V}\text{-CatCH}$ has equalisers and is therefore complete.

COALGEBRAS FOR THE HAUSDORFF FUNCTOR

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Theorem

The category of coalgebras of a Hausdorff polynomial functor on $\mathcal{V}\text{-Cat}\mathbf{CH}$ is (co)complete.

Definition

We call a functor **Hausdorff polynomial** whenever it belongs to the smallest class of endofunctors on $\mathcal{V}\text{-Cat}$ that contains the identity functor, all constant functors and is closed under composition with H , products and sums of functors.

Recall

An ordered compact Hausdorff space is a **Priestley** space whenever the cone $\mathbf{PosComp}(X, \mathbf{2})$ is an initial monocone. ^{ab}

^aHilary A. Priestley. “Representation of distributive lattices by means of ordered stone spaces”. In: *Bulletin of the London Mathematical Society* **2**.(2) (1970), pp. 186–190.

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Assumption

From now on we assume that the maps $\text{hom}(u, -): \mathcal{V} \rightarrow \mathcal{V}$ are continuous.

Definition

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Let X be a \mathcal{V} -categorical compact Hausdorff space. Consider a \mathcal{V} -subcategory $\mathcal{R} \subseteq \mathcal{V}^X$ that is closed under finite weighted limits and such that $(\psi: X \rightarrow \mathcal{V}^{\text{op}})_{\psi \in \mathcal{R}}$ is initial with respect to $\mathcal{V}\text{-CatCH} \rightarrow \text{CompHaus}$.

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Corollary

The Hausdorff functor restricts to a functor $\mathbb{H} : \mathcal{V}\text{-Priest} \rightarrow \mathcal{V}\text{-Priest}$, hence the Hausdorff monad \mathbb{H} restricts to $\mathcal{V}\text{-Priest}$.

DUALITY THEORY

Theorem

$\text{CoAlg}(\mathbf{H}) \simeq \mathbf{DLO}^{\text{op}}$ (*distributive lattices with operator*).^{ab}

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SOME CLASSICAL RESULTS

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$C: \mathbf{StablyComp}_{\mathbb{V}} \longrightarrow \mathbf{LaxMon}([0, 1]\text{-}\mathbf{FinSup})^{\text{op}}$ is fully faithful.^a

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“IRREDUCIBLE” DISTRIBUTORS

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^aMaria Manuel Clementino and Dirk Hofmann. “Lawvere completeness in topology”. In: *Applied Categorical Structures* **17**.(2) (2009), pp. 175–210.

^bDirk Hofmann and Isar Stubbe. “Towards Stone duality for topological theories”. In: *Topology and its Applications* **158**.(7) (2011), pp. 913–925.

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$[0, 1]$ is a Girard quantale: for every $u \in [0, 1]$, $u = u^{\perp\perp}$, $\text{hom}(u, \perp) = 1 - u =: u^{\perp}$.
Furthermore, the diagram

$$\begin{array}{ccccc} CX & \hookrightarrow & \mathbf{Fun}(X, [0, 1]^{\text{op}}) & \xrightarrow{(-)^{\perp}} & \mathbf{Fun}(X, [0, 1]^{\text{op}}) \\ & \searrow \Phi & \downarrow (-\cdot\varphi) & & \downarrow [\varphi, -]^{\text{op}} \\ & & [0, 1] & \xrightarrow{(-)^{\perp}} & [0, 1]^{\text{op}} \end{array}$$

commutes in $[0, 1]$ -Cat and $CX \hookrightarrow \mathbf{Fun}(X, [0, 1]^{\text{op}})$ is \vee -dense.

Conclusion: $\varphi: 1 \oplus X$ is left adjoint $\iff \Phi$ preserves finite weighted limits.