Duality, definability and conceptual completeness for κ-pretoposes

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The completeness theorem

Definition

A κ-topos is a topos of sheaves on a site with κ-small limits in which the covers of the topology satisfy in addition the transfinite transitivity property (a transfinite version of the transitivity property), i.e., transfinite composites of covering families form a covering family.

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A κ-point of a κ-topos is a point whose inverse image preserves all κ-small limits.

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Theorem

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$$\frac{\begin{array}{ll} \phi_f \vdash_{\mathbf{y}_f} \bigvee_{g \in \gamma^{\beta+1}, g|_\beta = f} \exists \mathbf{x}_g \phi_g & \beta < \kappa, f \in \gamma^\beta \\ \phi_f \dashv\vdash_{\mathbf{y}_f} \bigwedge_{\alpha < \beta} \phi_{f|_\alpha} & \beta < \kappa, \text{ limit } \beta, f \in \gamma^\beta \end{array}}{\phi_\emptyset \vdash_{\mathbf{y}_\emptyset} \bigvee_{f \in B} \exists_{\beta < \delta_f} \mathbf{x}_{f|_{\beta+1}} \bigwedge_{\beta < \delta_f} \phi_{f|_{\beta+1}}}$$

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$$\top \vdash_{xy} x < y \vee y < x \vee x = y$$

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López-Escobar: the theory of well-orderings is not axiomatizable in $\mathcal{L}_{\kappa,\omega}$ for any κ.

The λ-classifying topos of a κ-theory

Every κ-geometric theory has a κ-classifying topos:

$$\begin{tikzcd} \mathcal{C}_{\mathbb{T}} \arrow[rr, hook, "y"] \arrow[dr, "M"'] & & \mathcal{S}h(\mathcal{C}_{\mathbb{T}}, \tau) \arrow[dl, "f^*"] \\ & \mathcal{E} & \end{tikzcd}$$

Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$ (resp. κ is weakly compact).

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Let $\mathbb{T}'$ be the theory in $\mathcal{L}_{\lambda^+,\lambda}$ with the same axioms as those of $\mathbb{T}$.

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Theorem

(E.) The λ-classifying topos of $\mathbb{T}'$ is equivalent to the presheaf topos $\mathcal{S}et^{Mod_\lambda(\mathbb{T})}$. Moreover, the canonical embedding of the syntactic category

$$\mathcal{C}_{\mathbb{T}'} \hookrightarrow \mathcal{S}et^{Mod_\lambda(\mathbb{T})}$$

is given by the evaluation functor, which on objects acts by sending $(\mathbf{x}, \phi)$ to the functor $\{M \mapsto [[\phi]]^M\}$.

The λ-classifying topos of a κ-theory

The first consequence is a positive result regarding definability theorems for infinitary logic. If $\mathcal{C}_{\mathcal{T}}$ is the syntactic category of $\mathcal{T}$ considered in $\mathcal{L}_{\lambda^+,\lambda}$, we have that

$$ev : \mathcal{C}_{\mathcal{T}} \longrightarrow \mathcal{S}et^{Mod_{\lambda}(\mathcal{T})}$$

can be identified with Yoneda embedding

$$Y : \mathcal{C}_{\mathcal{T}} \longrightarrow \mathcal{S}h(\mathcal{C}_{\mathcal{T}}, \tau)$$

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Theorem

(Infinitary Beth) Let $\phi(R)$ be a formula in $\mathcal{L}_{\kappa^+,\kappa}$ over the language $\mathcal{L} \cup R$ containing the predicate R. If every $\mathcal{L}$-structure has a unique expansion to a model of $\phi(R)$ and the interpretation of R in each such model is preserved by $\mathcal{L}$-homomorphisms, then there is an $\mathcal{L}$-formula ψ of $\mathcal{L}_{\lambda^+,\lambda}$ such that $R \dashv\vdash_{\mathbf{x}} \psi$.

The λ-classifying topos of a κ-theory

Another consequence is the conceptual completeness theorem for $\mathcal{L}_{\kappa^+,\kappa}$:

Theorem

(Infinitary conceptual completeness) If a λ^+-coherent functor $I : \mathcal{P} \longrightarrow \mathcal{S}$, where $\mathcal{P}$ is a λ^+-pretopos, induces an equivalence between their categories of models $I^* : Mod(\mathcal{S}) \longrightarrow Mod(\mathcal{P})$, then I is itself an equivalence.

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(Infinitary Joyal) If $\mathcal{T}$ is intuitionistic first-order, the functor:

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This version of Joyal's theorem provides a proof of completeness with respect to Kripke models for theories in $\mathcal{L}_{\kappa^+,\kappa,\kappa}$.

Duality and descent

Consider a λ-accessible category K and the subcategory $Pres_\lambda(K)$ of its λ-presentable objects. Then the category of λ-points of the presheaf topos $\mathcal{S}et^{Pres_\lambda(K)}$ is equivalent to K itself.

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Definition

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between λ-accessible categories is λ-coherent if the induced functor $F^* : FC_\lambda(D, \mathcal{S}et) \longrightarrow FC_\lambda(C, \mathcal{S}et)$ preserves λ-coherent objects.

Duality and descent

Theorem

(Infinitary Stone duality) Let $\lambda > \kappa$ be weakly compact. There is a (bi-)equivalence (given by homming into $\mathcal{S}et$) between the following categories:

1. *$\mathcal{A}$: λ-pretopos completion of (syntactic categories of) theories in $\mathcal{L}_{\lambda,\lambda}$ with less than λ axioms; λ-pretopos morphisms; natural transformations.*
2. *$\mathcal{B}$: μ-accessible categories for $\mu < \lambda$; λ-accessible, λ-coherent functors preserving λ-presentable objects; natural transformations.*

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Corollary

The category of $\mathcal{A}$-morphisms between two objects $\mathcal{T}$ and $\mathcal{S}$, in $\mathcal{A}$, is equivalent to the category of $\mathcal{B}$-morphisms between Mod$(\mathcal{S})$ and Mod$(\mathcal{T})$.

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Theorem

(Infinitary Zawadowski) If κ is strongly compact, conservative κ-pretopos morphisms between κ-pretopose are of effective descent.

Categoricity and the λ-classifying topos

The completeness theorem allows to generalize a result of Barr and Makkai on the classifying topos of categorical theories:

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$$\mathbb{T}_\lambda := \mathbb{T} \cup \{\text{“}\textit{there are } \lambda \textit{ distinct elements}\text{”}\}$$

is two-valued and Boolean (alternatively, atomic and connected).

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Corollary

A κ-separable κ-topos has a unique point of cardinality at most κ (up to isomorphism) if and only if it is two-valued and Boolean (alternatively, atomic and connected).

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Shelah's eventual categoricity conjecture

Shelah's conjecture is an infinitary version of the behaviour of models of uncountable categorical theories:

Theorem

(Morley) If a countable theory $\mathbb{T}$ is categorical in an uncountable cardinal λ, then it is categorical in every uncountable cardinal λ.

Shelah extended this theorem to the case of uncountable theories and conjectured that, more generally, an eventual version holds for models of theories in $\mathcal{L}_{\omega_1,\omega}$ and even more general classes of models known as abstract elementary classes:

Conjecture

(Shelah) If a theory in $\mathcal{L}_{\omega_1,\omega}$ is categorical in a sufficiently high cardinal $\lambda \geq \kappa$, then it is categorical in all $\lambda \geq \kappa$.

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\begin{tikzcd}
\mathbf{Set}[\mathbb{T}_\lambda]_{\lambda^+} \cong \mathbf{Set}^{M_0^{\curvearrowright}} \arrow[d] & \mathbf{Set}[\mathbb{T}^1]_{\lambda^{++}} \cong \mathbf{Set}^{M_1^{\curvearrowright}} \arrow[dd, bend left] \\
\mathbf{Set}[\mathbb{T}_{\lambda^+}]_{\lambda^+} \arrow[r] \arrow[d] & \mathbf{Set}[\mathbb{T}_{\lambda^{++}}]_{\lambda^{++}} \arrow[d] \\
\mathbf{Set}[\mathbb{T}^1]_{\lambda^+} \cong \mathcal{S}h(M_0^{\curvearrowright op}, \tau_D) \arrow[r] \arrow[d, "M_1"'] & \mathbf{Set}[\mathbb{T}^1_{\lambda^{++}}]_{\lambda^{++}} \arrow[d] \\
\mathbf{Set} & \mathbf{Set}[\mathbb{T}^2]_{\lambda^{++}} \cong \mathcal{S}h(M_1^{\curvearrowright op}, \tau_D^1) \arrow[d, "M_2"'] \\
 & \mathbf{Set}
\end{tikzcd}
$$

Thank you!