What is a monoid?

How I learnt to stop worrying and love skewness

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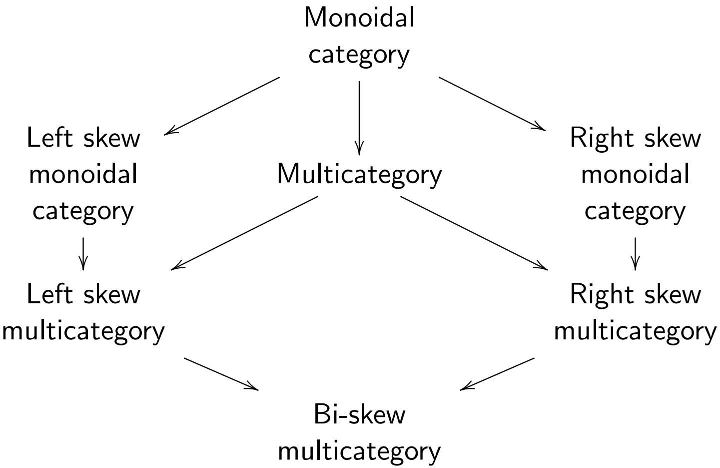
The big picture



The notion of monoid can be defined in each of these settings.

Outline

Monoid in a monoidal category

A monoid consists of

- an object a, the carrier
- a map $e\colon 1 \to a$, the unit
- a map $m\colon a \otimes a \to a$, the multiplication

Three diagrams must commute:

- Associativity
- Left unitality
- Right unitality.

Examples

- Monoid = monoid in $\mathbf{Set}$.
- Ring = monoid in $\mathbf{Ab}$.
- Algebra = monoid in $\mathbf{Vect}_{\mathbb{R}}$.
- Quantale = monoid in $\mathbf{CompSupLatt}$.
- Regular* cardinal = monoid in $\mathbf{Card}$.
- Monad on $\mathcal{C}$ = monoid in $[\mathcal{C}, \mathcal{C}]$.

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- Monad on $\mathcal{C}$ $=$ monoid in $[\mathcal{C}, \mathcal{C}]$.
- Monad on an object c of a bicategory.

Multicategory

In a multicategory, a morphism ("multi-map") goes from a list of objects to an object.

$$f\colon \overrightarrow{a} \to b$$

Example

Vector spaces and multilinear maps.

We have an identity maps $\mathrm{id}_a\colon a \to a$

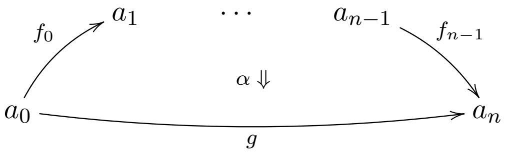
and can compose $f\colon \overrightarrow{a} \to b_i$ with $g\colon \overrightarrow{b} \to c$.

Four equations must be satisfied.

Virtual bicategories

A virtual bicategory has

- objects
- morphisms—not composable
- 2-cells



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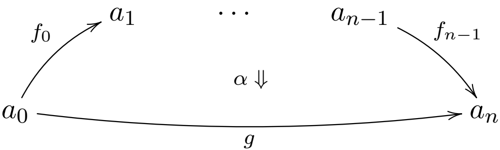


Also: virtual double categories.

Monoids and monads, using multi-maps

Monoid in a multicategory

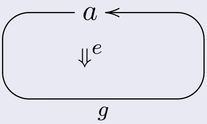
A monoid consists of an object a and multi-maps

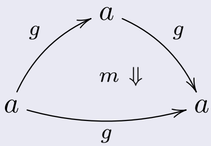
$$e\colon \ \to a \qquad\qquad m\colon a, a \to a$$

satisfying associativity, left and right unitality.

Monad on an object of a virtual bicategory

A monad on a consists of a 1-cell $g\colon a \to a$ and 2-cells





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Light category = monad in ?

Answer

The virtual bicategory of classes and set-valued relations.

A set-valued relation $A \nrightarrow B$ is a family of sets $(\mathcal{C}(a, b))_{a \in A, b \in B}$.

Composites don't exist; they would be class-valued.

Example: bimodules

Let $\mathbf{Bimod}$ be the virtual bicategory of light categories and bimodules.

A (light) bimodule $\mathcal{C} \nrightarrow \mathcal{D}$ is a functor $\mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathbf{Set}$.

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We can adapt this example to include strength. (Freyd category)

Multicategories vs monoidal categories

In some multicategories, tensors don't exist.

In others they exist but are complicated,

Compare:

- A quantale is a monoid in the monoidal category $\mathbf{CompSupLatt}$.
- A quantale is a monoid in the multicategory $\mathbf{CompSupLatt}$.

The latter is easy to unpack.

Skew monoidal categories (Szlachanyi)

A left skew monoidal category consists of

- a category $\mathcal{C}$
- an object 1
- a bifunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- an associator $(a \otimes b) \otimes c \to a \otimes (b \otimes c)$
- a left unitor $1 \otimes c \to c$
- a right unitor $a \to a \otimes 1$

satisfying five coherence laws.

Monoid in a skew monoidal category

In a skew monoidal category,

we can define monoids just as in a monoidal category.

Example: relative monads

Under certain size conditions:

relative monads are monoids in a skew monoidal category.

(Altenkirch, Chapman, Uustalu)

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Bourke and Lack introduced skew multicategories.

Skew multicategories

Bourke, Lack; Veltri, Uustalu, Zeilberger

In a left skew multicategory $\mathcal{C}$, a morphism goes from $s[\overrightarrow{a}$

where the house $\overrightarrow{a}$ is a list of objects.

and the left stoup s is either nothing or an object.

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When left-housing is invertible, $\mathcal{C}$ is "just" a multicategory.

Bi-skew multicategories

In a bi-skew multicategory, a morphism goes from $s[\overrightarrow{a}]t$.

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3 kinds of composition, 3 kinds of identity.

Monoid in a bi-skew multicategory

A monoid consists of

an object a and multi-maps $e\colon [] \to a$ and $m\colon a[]a \to a$

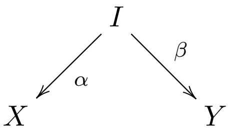
Associativity $a[a]a \to a$

Left unitality $a[] \to a$.

Right unitality $[]a \to a$.

Category on a span of classes

A (light) category on a span of classes



consists of the following:

- For each $x \in X, y \in Y$, a set $\mathcal{C}(x, y)$ of morphisms $x \to y$.
- For each $i \in I$, an identity $\mathrm{id}_i \colon \alpha(i) \to \beta(i)$.
- For each $f \colon x \to \beta(i)$ and $g \colon \alpha(i) \to y$, a composite $f; g \colon x \to y$.

Equations:

- Left identity for $g \colon \alpha(i) \to y$.
- Right identity for $f \colon x \to \beta(i)$.
- Associativity for $f \colon x \to \beta(i)$ and $g \colon \alpha(i) \to \beta(j)$ and $h \colon \alpha(j) \to y$.

Category on the span = monoid in ?

Answer: a bi-skew multicategory

An object is a family of sets $(\mathcal{A}(x,y))_{x \in X, y \in Y}$

A map $\mathcal{A}[\mathcal{B}]\mathcal{C} \to \mathcal{D}$ is a family of functions

$$\mathcal{A}(x, \beta(i)) \times \mathcal{B}(\alpha(i), \beta(j)) \times \mathcal{C}(\beta(j), y) \to \mathcal{D}(x, y)$$

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Relative monad

Let $\mathcal{O}$ be a bimodule $\mathcal{A} \nrightarrow \mathcal{B}$.

For example $\mathbf{FinSet} \nrightarrow \mathbf{Set}$ giving function sets.

A relative monad on $\mathcal{O}$ provides

- for each $a \in \mathcal{A}$, an object $Ta \in \mathcal{B}$ and unit $\eta_a \colon a \to Ta$
- for each $\mathcal{O}$-map $f \colon a \to Tb$, a $\mathcal{D}$-map $f^* \colon Ta \to Tb$

subject to the three "Kleisli triple" laws.

Relative monad = monoid in ?

Answer: a left skew multicategory

An object is a function $|\mathcal{A}| \to |\mathcal{B}|$.

A map $S[T_0, T_1] \to U$ is a family of maps

$$\mathcal{O}(a_0, T_0 a_1) \times \mathcal{O}(a_1, T_1 a_2) \to \mathcal{D}(S a_0, U a_2)$$

A map $[T_0, T_1] \to U$ is a family of maps

$$\mathcal{O}(a_0, T_0 a_1) \times \mathcal{O}(a_1, T_1 a_2) \to \mathcal{O}(a_0, U a_2)$$

Call-by-push-value: the type constructor F

- Two kinds of terms: values (e.g. variables) and computations.
- Two kinds of type: value type A and computation type $\underline{B}$.
- FA is the type of computations that aim to return a value of type A.

$$\frac{\Gamma \vdash^{\mathsf{v}} V : A}{\Gamma \vdash^{\mathsf{c}} \mathtt{return}\ V : FA} \qquad \frac{\Gamma \vdash^{\mathsf{c}} M : FA \quad \Gamma, x : A \vdash^{\mathsf{c}} N : \underline{B}}{\Gamma \vdash^{\mathsf{c}} M\ \mathtt{to}\ x.\ N : \underline{B}}$$

Three laws

$$\begin{array}{rcl} (M\ \mathtt{to}\ x.\ N)\ \mathtt{to}\ y.\ P & = & M\ \mathtt{to}\ x.\ (N\ \mathtt{to}\ y.\ P) \\ (\mathtt{return}\ V)\ \mathtt{to}\ x.\ M & = & M[V/x] \\ M & = & M\ \mathtt{to}\ x.\ \mathtt{return}\ x \end{array}$$

Conclusion

- In any bi-skew multicategory $\mathcal{C}$, we have a notion of a monoid.

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- By choosing an appropriate bi-skew multicategory, the following notions are monoid notions:
  - category on a given span of classes
  - model of the F fragment of call-by-push-value
  - relative monad on a bimodule
  - guardedness predicate.

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THANKS FOR LISTENING!