Intrinsic Schreier split extensions

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- [P] 2nd cohomology monoids $\leftrightarrow$ Schreier exts in **Mon**

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$\rightsquigarrow$ Study Schreier (split) extensions categorically

$\mathscr{S}$-protomodularity

- **Gp :** protomodular [B] **Mon :** non-protomodular

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- Ex: $\mathscr{S}$ = Schreier split epis of Jónsson–Tarski algebras $(\boldsymbol{x + 0 = x = 0 + x})$

  Any Jónsson–Tarski variety $\mathbb{V}$ is $\mathscr{S}$-protomodular

- This categorical approach for $\mathscr{S}$-protomodularity is not so categorical for $\mathscr{S}$ = Schreier split epis in **Mon** or Jónsson–Tarski variety $\mathbb{V}$

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$$\boldsymbol{K} \rhd\!\xrightarrow[\boldsymbol{k}]{} (\boldsymbol{X}, +, \boldsymbol{0}) \underset{\boldsymbol{f}}{\overset{\boldsymbol{s}}{\leftrightarrows}} \boldsymbol{Y}$$ Schreier split epi

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$$\begin{tikzcd} K \arrow[r, tail, "k"'] & \overset{\text{non-commutative}}{(X, +, 0)} \arrow[l, dashed, bend right, "\exists\, q"'] \arrow[r, two heads, shift right, "f"'] & Y \arrow[l, shift right, "s"'] \end{tikzcd}$$ Schreier split epi

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(S1) $x = kq(x) + sf(x)$, $\forall x \in X$

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(recover **Mon**/$\mathbb{V}$)

Imaginary morphisms - I

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  - $(\boldsymbol{P} \colon \mathbb{C} \to \mathbb{C}, \boldsymbol{\delta} \colon \boldsymbol{P} \Rightarrow \boldsymbol{P^2}, \boldsymbol{\varepsilon} \colon \boldsymbol{P} \Rightarrow \mathbf{1}_{\mathbb{C}})$ comonad

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- Def. An imaginary morphism from X to Y, denoted $X \dashrightarrow Y$, is a **real morphism** $P(X) \to Y$

$$\begin{tikzcd} K \arrow[r, tail, "k"'] & X \arrow[r, shift right, two heads, "f"'] & Y \arrow[l, shift right, "s"'] \\ & P(X) \arrow[ul, "q"] & \end{tikzcd}$$

in $\mathbb{C}$

$P(X)$ imaginary (Schreier) retraction

Imaginary morphisms - II

- $X \xrightarrow{f} Y$ real $\rightsquigarrow$ $X \overset{\overline{f}}{\dashrightarrow} Y$ imaginary $\left(P(X) \overset{\varepsilon_X}{\twoheadrightarrow} X \xrightarrow{f} Y \right)$

- $X \xrightarrow{f} Y$ real $\rightsquigarrow$ $X \overset{\overline{f}}{\dashrightarrow} Y$ imaginary $\quad (\; P(X) \overset{\varepsilon_X}{\twoheadrightarrow} X \xrightarrow{f} Y \;)$

  $Y \xrightarrow{1_Y} Y$ real $\rightsquigarrow$ $Y \overset{\overline{1_Y}}{\dashrightarrow} Y$ imaginary $\quad (\; P(Y) \overset{\varepsilon_Y}{\twoheadrightarrow} Y \;)$

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- $$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[rr, dashed, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd} \rightsquigarrow P(X) \xrightarrow{f} Y \xrightarrow{g} Z$$

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  $$\begin{tikzcd} W \arrow[r, "h"] \arrow[rr, dashed, bend right, "f\circ h"'] & X \arrow[r, dashed, "f"] & Y \end{tikzcd} \quad \rightsquigarrow \quad P(W) \xrightarrow{P(h)} P(X) \xrightarrow{f} Y$$

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  $$\begin{tikzcd} Y \arrow[r, dashed, "s"] \arrow[rr, dashed, bend right, "f\circ s=\overline{1_Y}"'] & X \arrow[r, "f"] & Y \end{tikzcd} \quad \left(\begin{tikzcd} P(Y) \arrow[r, "s"] \arrow[rr, bend right, twoheadrightarrow, "fs=\varepsilon_Y"'] & X \arrow[r, "f"] & Y \end{tikzcd} \right)$$

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  iff $\forall r_{A,B} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \colon A + B \twoheadrightarrow A \times B$ regular epi

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  iff $\exists$ imaginary splitting
  (+ projs)

$$\begin{tikzcd} P(A \times B) \arrow[r, "\exists\, t_{A,B}"] \arrow[rr, bend right, "(*)", "r_{A,B} t_{A,B} = \varepsilon_{A \times B}"'] & A + B \arrow[r, two heads, "r_{A,B}"] & A \times B \end{tikzcd}$$

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- $\mathbb{V}$ Jónsson–Tarski variety $\rightsquigarrow$ $\exists$ $t_{A,B} \colon P(A \times B) \to A + B$

$$\begin{aligned} t_{A,B} \colon P(A \times B) &\to A + B \\ [(a,b)] &\mapsto \underline{a} + \overline{b} \end{aligned}$$

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  iff $\forall r_{A,B} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) : A + B \twoheadrightarrow A \times B$ regular epi

  iff $\exists$ imaginary splitting (+ projs)

$$\begin{tikzcd} P(A \times B) \arrow[r, "\exists\, t_{A,B}"] \arrow[rr, bend right, "(*)", "r_{A,B} t_{A,B} = \varepsilon_{A \times B}"'] & A + B \arrow[r, two heads, "r_{A,B}"] & A \times B \end{tikzcd}$$

- $\mathbb{V}$ Jónsson–Tarski variety $\rightsquigarrow$ $\exists$ $t_{A,B} : P(A \times B) \to A + B$

$$[(a, b)] \mapsto \underline{a} + \overline{b}$$

natural transformation

- **Mon** unital category $\rightsquigarrow$ Jónsson–Tarski variety $(\, x + 0 = x = 0 + x \,)$

- $\mathbb{C}$ pointed + regular + binary coproducts is unital

  iff $\forall r_{A,B} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \colon A + B \twoheadrightarrow A \times B$ regular epi

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- $\mathbb{V}$ Jónsson–Tarski variety $\rightsquigarrow$ $\exists$

$$t_{A,B} \colon \begin{array}{ccc} P(A \times B) & \to & A + B \\ [(a, b)] & \mapsto & \underline{a} + \overline{b} \end{array}$$

natural transformation

- natural imaginary splitting: $t \colon P((\cdot) \times (\cdot)) \Rightarrow (\cdot) + (\cdot)$ sth $(*)$ in $\mathbb{C}$

Imaginary addition - I

- $t \rightsquigarrow \mu^X : X \times X \dashrightarrow X$ natural imaginary addition

$$P(X \times X) \xrightarrow{t_{X,X}} X + X \xrightarrow{(1\ \ 1)} X$$

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- $$\begin{tikzcd}
X \arrow[dr, "\langle 1,0 \rangle"'] \arrow[drr, dashed, bend left, "\mu^X \circ \langle 1,0 \rangle = \overline{1_X}"] & & \\
& X \times X \arrow[r, dashed, "\mu^X"] & X \\
X \arrow[ur, "\langle 0,1 \rangle"] \arrow[urr, dashed, bend right, "\mu^X \circ \langle 0,1 \rangle = \overline{1_X}"'] & &
\end{tikzcd}$$

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(pps of t)

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(pps of t)

- $\forall f : X \to Y,$

$$\begin{tikzcd} X \times X \arrow[r, dashed, "\mu^X"] \arrow[d, "f \times f"'] & X \arrow[d, "f"] \\ Y \times Y \arrow[r, dashed, "\mu^Y"'] & Y \end{tikzcd}$$

$$f \circ \mu^X = \mu^Y \circ (f \times f)$$

- $t \rightsquigarrow \mu^X : X \times X \dashrightarrow X$ natural imaginary addition

$$P(X \times X) \xrightarrow{t_{X,X}} X + X \xrightarrow{(1\ \ 1)} X$$

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$$f \circ \mu^X = \mu^Y \circ (f \times f)$$

(naturality of t)

Imaginary addition - II

- $A \xrightarrow{g} X \xleftarrow{h} A$

  $g(a) + h(a)$

  $A \xrightarrow{\langle g,h \rangle} X \times X \dashrightarrow^{\mu^X} X$

Imaginary addition - II

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$$\begin{tikzcd} P(A) \arrow[d, "P\langle g,h\rangle"'] & & \\ P(X \times X) \arrow[r, "t_{X,X}"'] & X + X \arrow[r, "(1\ \ 1)"'] & X \end{tikzcd}$$

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P(A) \arrow[r, "P\langle 1,1 \rangle"] \arrow[d, "P\langle g,h \rangle"'] & P(A \times A) \arrow[dl, "P(g \times h)"] & \\
P(X \times X) \arrow[r, "t_{X,X}"'] & X + X \arrow[r, "(1 \ \ 1)"'] & X
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P(X \times X) \arrow[r, "t_{X,X}"'] & X + X \arrow[r, "(1\ \ 1)"'] & X
\end{tikzcd}$$

- $A \xrightarrow{g} X \xleftarrow{h} A$

$$\color{red}{g(a) + h(a)}$$

$$A \xrightarrow{\langle g,h \rangle} X \times X \overset{\mu^X}{\dashrightarrow} X$$

$$\begin{tikzcd}
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P(X \times X) \arrow[r, "t_{X,X}"'] & X + X \arrow[r, "(1\ \ 1)"'] & X
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P(X \times X) \arrow[rr, "t_{X,X}"'] & & X + X \arrow[r, "(1\ \ 1)"'] & X
\end{tikzcd}$$

- $A \xrightarrow{g} X \xleftarrow{j} B$

  $g(a) + j(b)$

  $A \times B \xrightarrow{g \times j} X \times X \overset{\mu^X}{\dashrightarrow} X$

- $A \xrightarrow{g} X \xleftarrow{h} A$

  $g(a) + h(a)$

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\end{tikzcd}$$

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P(X \times X) \arrow[r, "t_{X,X}"'] & X + X \arrow[r, "\begin{pmatrix} 1 & 1 \end{pmatrix}"'] & X
\end{tikzcd}$$

Intrinsic Schreier split epis

- $\mathbb{C}$ regular unital category w/ binary coproducts, functorial projective covers and natural imaginary splitting t

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- $$\begin{tikzcd} K \arrow[r, tail, "k"'] & X \arrow[r, shift right, two heads, "f"'] & Y \arrow[l, shift right, "s"'] \end{tikzcd}$$ intrinsic Schreier split epi

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- $\mathbb{C}$ regular unital category w/ binary coproducts, functorial projective covers and natural imaginary splitting t

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(iS1)

$$x \overset{(\mathrm{S1})}{=} kq(x) + sf(x)$$

Intrinsic Schreier split epis

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- $$\begin{tikzcd} K \arrow[r, "k"', tail] & X \arrow[l, dashed, bend right, "\exists\, q"'] \arrow[r, "f"', two heads, shift right] & Y \arrow[l, "s"', shift right] \end{tikzcd}$$ intrinsic Schreier split epi

(iS1)
$$\begin{tikzcd} P^2(X) \arrow[r, "P\langle 1,1\rangle"] & P(P(X)\times P(X)) \arrow[r, "t_{P(X),P(X)}"] & P(X)+P(X) \arrow[d, "(kq \;\; sf\varepsilon_X)"] \\ & & X \end{tikzcd}$$

$$x \overset{(\mathrm{S1})}{=} kq(x) + sf(x)$$

- $\mathbb{C}$ regular unital category w/ binary coproducts, functorial projective covers and natural imaginary splitting t

- $\begin{tikzcd} K \arrow[r, tail, "k"'] & X \arrow[l, dashed, bend right, "\exists\, q"'] \arrow[r, two heads, shift right, "f"'] & Y \arrow[l, shift right, "s"'] \end{tikzcd}$ intrinsic Schreier split epi

(iS1)
$$\begin{tikzcd} P^2(X) \arrow[r, "P\langle 1,1\rangle"] & P(P(X) \times P(X)) \arrow[r, "t_{P(X),P(X)}"] & P(X) + P(X) \arrow[d, "(kq \;\; sf\varepsilon_X)"] \\ P(X) \arrow[rr, two heads, "\varepsilon_X"'] & & X \end{tikzcd}$$

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- $$\begin{tikzcd} K \arrow[r, tail, "k"'] & X \arrow[l, dashed, bend right, "\exists\, q"'] \arrow[r, two heads, shift right, "f"'] & Y \arrow[l, shift right, "s"'] \end{tikzcd}$$ intrinsic Schreier split epi

(iS1)
$$\begin{tikzcd} P^2(X) \arrow[r, "P\langle 1,1\rangle"] & P(P(X)\times P(X)) \arrow[r, "t_{P(X),P(X)}"] & P(X)+P(X) \arrow[d, "(kq\ \ sf\varepsilon_X)"] \\ P(X) \arrow[u, "\delta_X"] \arrow[rr, two heads, "\varepsilon_X"'] & x \overset{(S1)}{=} kq(x)+sf(x) & X \end{tikzcd}$$

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- $$\begin{tikzcd} K \arrow[r, tail, "k"'] & X \arrow[l, dashed, bend right, "\exists\, q"'] \arrow[r, two heads, shift right, "f"'] & Y \arrow[l, shift right, "s"'] \end{tikzcd}$$ intrinsic Schreier split epi

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(iS2)

$$a \overset{(\mathrm{S2})}{=} q(k(a)+s(y))$$

Intrinsic Schreier split epis

- $\mathbb{C}$ regular unital category w/ binary coproducts, functorial projective covers and natural imaginary splitting t

- $K \underset{k}{\rightarrowtail} X \underset{f}{\overset{s}{\rightleftarrows}} Y$, $\exists\, q \colon X \dashrightarrow K$ intrinsic Schreier split epi

(iS1)

$$\begin{tikzcd}
P^2(X) \arrow[r, "P\langle 1{,}1\rangle"] & P(P(X)\times P(X)) \arrow[r, "t_{P(X),P(X)}"] & P(X)+P(X) \arrow[d, "(kq \;\; sf\varepsilon_X)"] \\
P(X) \arrow[u, "\delta_X"] \arrow[rr, two heads, "\varepsilon_X"'] \arrow[urr, phantom, "x \overset{(\mathrm{S1})}{=} kq(x)+sf(x)"] & & X
\end{tikzcd}$$

(iS2)

$$\begin{tikzcd}
P^2(K\times Y) \arrow[r, "P(t_{K,Y})"] & P(K+Y) \arrow[r, "P(k \;\; s)"] & P(X) \arrow[d, "q"] \\
& a \overset{(\mathrm{S2})}{=} q(k(a)+s(y)) & K
\end{tikzcd}$$

- $\mathbb{C}$ regular unital category w/ binary coproducts, functorial projective covers and natural imaginary splitting t

- $$\begin{tikzcd} K \arrow[r, tail, "k"'] & X \arrow[l, dashed, bend right, "\exists\, q"'] \arrow[r, two heads, shift right, "f"'] & Y \arrow[l, shift right, "s"'] \end{tikzcd}$$ intrinsic Schreier split epi

(iS1)
$$\begin{tikzcd} P^2(X) \arrow[r, "P\langle 1{,}1\rangle"] & P(P(X)\times P(X)) \arrow[r, "t_{P(X),P(X)}"] & P(X)+P(X) \arrow[d, "(kq\ \ sf\varepsilon_X)"] \\ P(X) \arrow[u, "\delta_X"] \arrow[rr, two heads, "\varepsilon_X"'] \arrow[rru, phantom, "x \overset{(S1)}{=} kq(x)+sf(x)"] & & X \end{tikzcd}$$

(iS2)
$$\begin{tikzcd} P^2(K\times Y) \arrow[r, "P(t_{K,Y})"] & P(K+Y) \arrow[r, "P(k\ \ s)"] & P(X) \arrow[d, "q"] \\ P(K\times Y) \arrow[r, two heads, "\varepsilon_{K\times Y}"'] \arrow[rru, phantom, "a \overset{(S2)}{=} q(k(a)+s(y))"] & K\times Y \arrow[r, two heads, "\pi_K"'] & K \end{tikzcd}$$

Intrinsic Schreier split epis

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- $$\begin{tikzcd} K \arrow[r, tail, "k"'] & X \arrow[l, dashed, bend right, "\exists\, q"'] \arrow[r, two heads, "f"'] & Y \arrow[l, bend right, "s"'] \end{tikzcd}$$ intrinsic Schreier split epi

(iS1)
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(iS2)
$$\begin{tikzcd} P^2(K\times Y) \arrow[r, "P(t_{K,Y})"] & P(K+Y) \arrow[r, "P(k\ \ s)"] & P(X) \arrow[d, "q"] \\ P(K\times Y) \arrow[u, "\delta_{K\times Y}"] \arrow[r, two heads, "\varepsilon_{K\times Y}"'] \arrow[urr, phantom, "a \overset{(\mathrm{S2})}{=} q(k(a)+s(y))"] & K\times Y \arrow[r, two heads, "\pi_K"'] & K \end{tikzcd}$$

- $q\colon X \dashrightarrow K$ imaginary Schreier retraction

  $qk = 1_K$ in **Mon** $\rightsquigarrow$ $q \circ k = \overline{1_K} \Leftrightarrow qP(k) = \varepsilon_K$ in $\mathbb{C}$

Properties - I

- $q\colon X \dashrightarrow K$ imaginary Schreier retraction

  $qk = 1_K$ in **Mon** $\rightsquigarrow$ $q \circ k = \overline{1_K} \Leftrightarrow qP(k) = \varepsilon_K$ in $\mathbb{C}$

- $qs = 0$ in **Mon** $\rightsquigarrow$ $qP(s) = 0$ in $\mathbb{C}$

Properties - I

- $q\colon X \dashrightarrow K$ imaginary Schreier retraction

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  $q(0) = 0$ in **Mon** $\rightsquigarrow$ obvious in $\mathbb{C}$

- $q\colon X \dashrightarrow K$ imaginary Schreier retraction
  $qk = 1_K$ in **Mon** $\rightsquigarrow$ $q \circ k = \overline{1_K} \Leftrightarrow qP(k) = \varepsilon_K$ in $\mathbb{C}$

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  $kq(s(y) + k(a)) + s(y) = s(y) + k(a)$ in **Mon** $\rightsquigarrow$ ✓ in $\mathbb{C}$

Properties - I

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  $qk = 1_K$ in **Mon** $\rightsquigarrow$ $q \circ k = \overline{1_K} \Leftrightarrow qP(k) = \varepsilon_K$ in $\mathbb{C}$

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  $kq(s(y) + k(a)) + s(y) = s(y) + k(a)$ in **Mon** $\rightsquigarrow$ ✓ in $\mathbb{C}$

- q is unique

- $q\colon X \dashrightarrow K$ imaginary Schreier retraction
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- $X \rhd\!\xrightarrow[\langle 1_X, 0\rangle]{} X \times Y \underset{\pi_Y}{\overset{\langle 0, 1_Y\rangle}{\rightleftarrows}} Y$ intrinsic Schreier split extension

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- [MRVdL] Y protomodular object: all points $X \leftrightarrows Y$ are stably strong

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$\rightsquigarrow$ If all $X \leftrightarrows Y$ satisfy **(iS1)**, then Y is a protomodular object

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  - intrinsic Schreier split epi wrt $\boldsymbol{t_{A,B} \colon P(A \times B) \to A + B}$

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Main results

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- Future: higher-order cohomology groups