# The (big) infinitesimal topos as a classifying topos

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**Goal:** Understand toposes from algebraic geometry (from a logical perspective).

**Promise:** You will fully understand the key ingredient of the proof (in a simplified case)!

# Toposes

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**Ex:** $\mathrm{Sh}(X)$ is a topos.

## Definition

A *site* is a small category $C$ together with a *Grothendieck topology* $J$, distinguishing some *covering* families $(c_i \to c)_{i \in I}$.
A *sheaf* is a presheaf $F : C^{\mathrm{op}} \to \mathbf{Set}$ satisfying a "glueing" condition for every covering family $(c_i \to c)_{i \in I}$.

## Definition

A *(Grothendieck) topos* is a category equivalent to some $\mathrm{Sh}(C, J)$.

# Geometric theories

A *geometric theory* consists of:

- sorts
- function symbols
- relation symbols
- axioms

The theory of  rings:

- one sort: $A$
- five function symbols: $0, 1 : A$, $+, \cdot : A \times A \to A$, $- : A \to A$
- no relation symbols
- eight axioms:

$$0 + x = x,$$
$$x \cdot y = y \cdot x,$$

$\ldots$

# Geometric theories

A *geometric theory* consists of:

- sorts
- function symbols
- relation symbols
- axioms $\phi \vdash \psi$,
  where $\phi$ and $\psi$ may contain $\top, \bot, \wedge, \vee, \bigvee, \exists$
  but no $\bigwedge, \forall, \Rightarrow, \neg$

The theory of *local* rings:

- one sort: $A$
- five function symbols:
  $0, 1 : A$, $+, \cdot : A \times A \to A$,
  $- : A \to A$
- no relation symbols
- eight axioms:
  $\top \vdash_x 0 + x = x$,
  $\top \vdash_{x,y} x \cdot y = y \cdot x$,
  $\ldots$,
  $0 = 1 \vdash \bot$,
  $x + y = 1 \vdash_{x,y}$
  $(\exists z.\ xz = 1) \vee (\exists z.\ yz = 1)$

# Classifying toposes

## Definition

A *classifying topos* for $\mathbb{T}$ is a topos $\mathbf{Set}[\mathbb{T}]$ with

$$\mathbb{T}(\mathcal{E}) \simeq \mathbf{Geom}(\mathcal{E}, \mathbf{Set}[\mathbb{T}])$$

for every topos $\mathcal{E}$.

In other words, there is a *universal model* of $\mathbb{T}$ in $\mathbf{Set}[\mathbb{T}]$.

## Theorem

*Every geometric theory has a classifying topos.*
*Every topos classifies some geometric theory.*

# Theories of presheaf type

## Definition

$\mathbb{T}$ is *of presheaf type* if $\mathbf{Set}[\mathbb{T}] \simeq [C^{\mathrm{op}}, \mathbf{Set}]$ for some $C$.

## Theorem

*Any algebraic theory is of presheaf type.*
*Any Horn theory (only $\top, \wedge$, no $\bot, \vee, \bigvee, \exists$) is of presheaf type.*
*Any cartesian theory is of presheaf type.*

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## Theorem

*If $\mathbb{T}$ is of presheaf type, then $\mathbf{Set}[\mathbb{T}] \simeq [\mathbb{T}(\mathbf{Set})_c, \mathbf{Set}]$,*
*where $-_c$ denotes the compact objects*
*(those for which $\mathrm{Hom}_{\mathbb{T}(\mathbf{Set})}(M, -)$ preserves filtered colimits).*

**Ex:** The theory of rings is classified by $[\mathbf{Ring}_c, \mathbf{Set}] = [\mathbf{Ring}_{fp}, \mathbf{Set}]$.
**Ex:** The *object classifier* is $[\mathbf{Set}_c, \mathbf{Set}] = [\mathbf{FinSet}, \mathbf{Set}]$.

additional axioms $\leftrightarrow$ subtopos $\leftrightarrow$ Grothendieck topology additional axioms $\leftrightarrow$ subtopos $\leftrightarrow$ Grothendieck topology

## Example

For $\mathbb{T}$ = theory of rings, the axioms

- $0 = 1 \ \vdash \ \bot$
- $x + y = 1 \ \vdash_{x,y} \ (\exists z.\ xz = 1) \vee (\exists z.\ yz = 1)$

mean:

- The zero-ring is covered by the empty family.
- $A$ is covered by $A[x^{-1}]$ and $A[y^{-1}]$ whenever $x + y = 1$.

## Corollary

*The (big) Zariski topos classifies the theory of local rings.*

# The infinitesimal topos (simple version)

## Definition

The *(big) infinitesimal topos* is $\mathrm{Sh}(C, J)$ with $C$, $J$ as follows.

$$\begin{tikzcd} \mathfrak{a} \arrow[r, hook] \arrow[d] & A \arrow[d] \\ \mathfrak{a}' \arrow[r, hook] & A' \end{tikzcd}$$

$C = \{$finitely presented rings $A$ with a finitely generated ideal $\mathfrak{a} \subseteq A$ such that every element of $\mathfrak{a}$ is nilpotent$\}^{\mathrm{op}}$

*Hey, this is the category of compact models of a geometric theory* $\mathbb{T}_{\mathrm{inf}}$*!*

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$J =$ Zariski topology on $C$.

*This will correspond to "local ring" axioms again.*

# The key ingredient: Is $\mathbb{T}_{\mathrm{inf}}$ of presheaf type?

$\mathfrak{a} \subseteq A$ with

$$\begin{aligned}
\top &\vdash 0 \in \mathfrak{a} \\
x \in \mathfrak{a} &\vdash_{x,y} x \cdot y \in \mathfrak{a} \\
x \in \mathfrak{a} \wedge y \in \mathfrak{a} &\vdash_{x,y} x + y \in \mathfrak{a} \\
x \in \mathfrak{a} &\vdash_{x} \bigvee_{n \in \mathbb{N}} x^n = 0
\end{aligned}$$

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$$x \in \mathfrak{a} \vdash_x \bigvee_{n \in \mathbb{N}} x^n = 0$$

$\mathfrak{a}_n \subseteq A$, for each $n \in \mathbb{N}$, with

$$x \in \mathfrak{a}_n \dashv\vdash_x x^n = 0 \wedge x \in \mathfrak{a}_{n+1}$$

$$\top \vdash 0 \in \mathfrak{a}_1$$

$$x \in \mathfrak{a}_n \vdash_{x,y} x \cdot y \in \mathfrak{a}_n$$

$$x \in \mathfrak{a}_n \wedge y \in \mathfrak{a}_n \vdash_{x,y} x + y \in \mathfrak{a}_{2n-1}$$

These theories are Morita equivalent!

# General case

Let $R$ be a finitely presented $K$-algebra.

## Theorem

*The big infinitesimal topos of* $\operatorname{Spec} R / \operatorname{Spec} K$ *classifies the theory of surjective* $K$*-algebra homomorphisms* $f : A \twoheadrightarrow B$ *into an* $R$*-algebra* $B$ *with locally nilpotent kernel.*

$$\begin{array}{ccc} K & \longrightarrow & R \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} \!\!\!\!\twoheadrightarrow & B \end{array}$$

$$\top \vdash_{y:B} \exists x : A.\ f(x) = y$$

$$f(x) = 0 \vdash_{x:A} \bigvee_{n \in \mathbb{N}} x^n = 0$$

**Proof idea:** Start with algebraic theory, $f : A \to B$. Show that the induced topology is *rigid*.

# Future work

- What about the crystalline topos? [Coming soon!]
- Can we apply this in algebraic geometry?

For more details see:

https://gitlab.com/MatthiasHu/master-thesis/raw/master/thesis.pdf