Profinite Monads and Reiterman's Theorem

J. Adámek, L.-T. Chen, S. Milius and H. Urbat

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The Birkhoff Variety Theorem (1935)

- **The Birkhoff Theorem**
  $\mathcal{A}$ a full subcategory of Σ-**Alg**:

  $\mathcal{A}$ presentable by equations
  $\Leftrightarrow$ variety (= HSP class)

  regular quotients subobjects products

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- Lawvere: equations are pairs of *n.t.* $\alpha\colon U^n \to U$ for

  $$U\colon \Sigma\text{-}\mathbf{Alg} \to \mathbf{Set}$$

  An algebra A satisfies $\alpha = \alpha'$ iff $\alpha_A = \alpha'_A$

The Reiterman Theorem (1982)

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- $U_f \colon (\Sigma\text{-}\mathbf{Alg})_f \to \mathbf{Set}_f$
  Pseudoequations are pairs of *n.t.* $\alpha \colon U_f^n \to U_f$
  a finite algebra A satisfies $\alpha = \alpha'$ iff $\alpha_A = \alpha'_A$

The Reiterman Theorem (1982)

- **Example** *Un*, unary algebras

$$\sigma\colon A \to A$$

A finite $\Rightarrow \exists\, n\colon \sigma^n = (\sigma^n)^2$
Notation : $\sigma^* = \sigma^n$

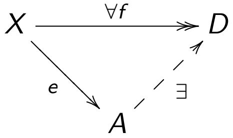
Pseudoequation : $\sigma^*(x) = x$
presents : finite algebras
with σ invertible

Banaschewski and Herrlich (1976)

- $\mathcal{D}$ a complete category
  $(\mathcal{E}, \mathcal{M})$ a proper factorization system (e.g. regular epi - mono)
  notation $\twoheadrightarrow$ and $\rightarrowtail$
  $\mathcal{D}$ has enough projectives $X\colon \forall\, D \exists\, X \twoheadrightarrow D$

  Definitions An equation $e\colon X \twoheadrightarrow A$, X projective.

  It is satisfied by $D \in \mathcal{D}$ if



  (D is e-injective)

  Theorem A full subcategory $\mathcal{A}$ of $\mathcal{D}$:
  $\mathcal{A}$ presentable by equations $\Leftrightarrow$ a variety ($=$ HSP class)

Pseudovariety Presentation

- Assume : $\mathcal{D}$ and $(\mathcal{E}, \mathcal{M})$ as above
  $\mathcal{D}_f \subseteq \mathcal{D}$ full subcategory closed under S and P_f
  'finite' objects

Pseudovariety Presentation

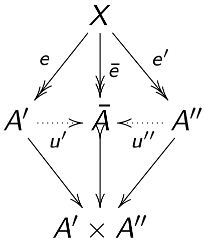
- Assume : $\mathcal{D}$ and $(\mathcal{E}, \mathcal{M})$ as above
  $\mathcal{D}_f \subseteq \mathcal{D}$ full subcategory closed under S and P_f
  'finite' objects
- **Definition** A **pseudovariety** is a full subcategory of $\mathcal{D}_f$ closed under HSP_f.

Pseudovariety Presentation

- **Definition** A **quasi-equation** over X (projective) is a semilattice Ω of finite quotients $e\colon X \twoheadrightarrow A$ $(A \in \mathcal{D}_f)$

$$\forall e,\ e' \in \Omega$$

$$\bar{e} = e \wedge e' \in \Omega$$



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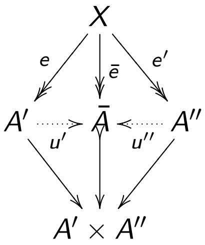
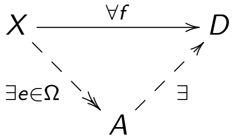


- An object D satisfies Ω if it is injective:



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- **Proposition** $\mathcal{A}$ a full subcategory of $\mathcal{D}_f$:
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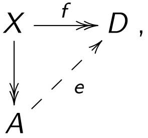
- Proof $\Leftarrow$ For every X projective

$$\Omega_X : \quad X \twoheadrightarrow \quad A (A \in \mathcal{A})$$

Ω_X semilattice $\Leftarrow$ $\mathcal{A}$ is SP_f-class
$D \in \mathcal{A} \Rightarrow D$ satisfies Ω_X ...trivial

D satisfies each $\Omega_X \Rightarrow D \in \mathcal{A}$: choose



X projective, $e \in \mathcal{E}$, $A \in \mathcal{A} \Rightarrow D \in \mathcal{A}$

Our Goal

- Given : $\mathcal{D}$, $(\mathcal{E}, \mathcal{M})$ and $\mathcal{D}_f$ as above
  $\mathbb{T}$ a monad on $\mathcal{D}$ preserving $\mathcal{E}$
  Describe pseudovarieties in $\mathcal{D}^{\mathbb{T}}$ by **equations**
  in some extension of $\mathcal{D}^{\mathbb{T}}$

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- $\mathcal{D}^{\mathbb{T}}$ has the factorization system inherited from $\mathcal{D}$
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  $\mathcal{D}_f^{\mathbb{T}} \overset{\text{def}}{=}$ all algebras (A, α) with $A \in \mathcal{D}_f$
- Thus pseudovarieties are presentable by quasi-equations in $\mathcal{D}^{\mathbb{T}}$

The Category $\hat{\mathcal{D}}_f$

- **Profinite completion** Pro $\mathcal{D}_f = \hat{\mathcal{D}}_f$ (dual to Ind)
  $\mathcal{D}_f$ finitely complete $\Rightarrow \hat{\mathcal{D}}_f$ complete
  $\hat{\mathcal{E}} =$ cofiltered limits of quotients in $\mathcal{D}_f$
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- **Wanted**: $\hat{\mathcal{D}}_f$ has enough $\hat{\mathcal{E}}$-projectives
  $\mathbb{T}$ yields (canonically) a monad $\hat{\mathbb{T}}$ on $\hat{\mathcal{D}}_f$ preserving $\hat{\mathcal{E}}$

$\Rightarrow \hat{\mathcal{D}}_f$, $(\hat{\mathcal{E}}, \hat{\mathcal{M}})$ and $\hat{\mathbb{T}}$ satisfy all of our assumptions

Goal : quasi-equations in $\mathcal{D}^{\mathbb{T}} \Leftrightarrow$ equations in $(\hat{\mathcal{D}}_f)^{\hat{\mathbb{T}}}$

Important: $\mathbb{T}$ and $\hat{\mathbb{T}}$ have the same finite algebras

$$\mathcal{D}_f^{\mathbb{T}} \simeq \hat{\mathcal{D}}_f^{\hat{\mathbb{T}}}$$

Profinite Factorization Systems

Definition $(\mathcal{E}, \mathcal{M})$ is a **profinite** factorization system if
$\mathcal{E}$ is closed under cofiltered limits of quotients in $\mathcal{D}_f^{\rightarrow}$

Examples with $\mathcal{E}$ = surjective morphisms

- **Set** : $\hat{\mathbf{Set}}_f$ = **Stone**

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  $\hat{\textbf{Pos}}_f$ = **Priestley**

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- **Set** : $\hat{\mathbf{Set}}_f = \mathbf{Stone}$
- **Pos** : with $\mathcal{E}$ = surjective monotone maps
  $\hat{\mathbf{Pos}}_f = \mathbf{Priestley}$
- $\mathcal{D} \subseteq \Sigma$-**Str** full subcategory closed under limits
  arbitrary operation symbols
  $+$ finitely many relation symbols

Pro $\mathcal{D}_f \subseteq$ Stone $\mathcal{D}$
$\hat{\mathcal{E}}$ = surjective continuous homomorphisms

Profinite monad $\hat{\mathbb{T}}$

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  $\hat{\mathbb{T}}$ is the monad of **profinite words** on $\hat{\mathsf{Mon}}_f = \mathsf{Stone\ Mon}$

- $\hat{T}Y$ is the cofiltered limit of all finite $\hat{\mathcal{E}}$-quotients of Y carried by $\mathbb{T}$-algebras

  Example For $TX = X^*$: a profinite word in a Stone monoid Y is a compatible choice of a member of A for every finite quotient monoid A of Y.

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- **Proposition** If $(\mathcal{E}, \mathcal{M})$ is profinite then
  (1) $\hat{\mathbb{T}}$ preserves $\hat{\mathcal{E}}$
  and
  (2) finite $\mathbb{T}$-algebras coincide with finite $\hat{\mathbb{T}}$-algebras

Generalized Reiterman's Theorem

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- **Theorem** $\mathcal{A}$ a full subcategory of $\mathcal{D}_f^{\mathbb{T}}$:
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Profinite equations in Σ-**Str**

- **Example** $\mathcal{D} \subseteq \Sigma$- **Str** closed under limits and subobjects
  $\hat{\mathcal{D}}_f \subseteq$ Stone Σ-**Str**

  A **profinite equation** : $\alpha = \alpha'$ where $\alpha, \alpha' \in \hat{T}X$

  X projective in $\hat{\mathcal{D}}_f$

  Given $e\colon (\hat{T}X, \mu_X) \twoheadrightarrow A$, take all $(\alpha, \alpha') \in \ker$

Profinite Equations in Σ-Str

- Back to Reiterman : $U_f\colon (\Sigma\text{-}\mathbf{Alg})_f \to \mathbf{Set}_f$

$$\text{n.t. } \alpha\colon U_f^n \to U_f \quad \leftrightsquigarrow \quad \text{elements of } \hat{T}n$$

$$\text{pseudoequations} \quad \leftrightsquigarrow \quad \text{profinite equations}$$

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- Varieties of ordered algebras . . . inequalities $\alpha \leq \alpha'$ between terms

$\mathcal{D} = \mathbf{Pos}$ $\quad \hat{\mathcal{D}}_f = \text{Priestley}$

profinite equations $e \colon (\hat{T}X, \mu_X) \twoheadrightarrow A$,

X discretely ordered

$\leftrightsquigarrow$ inequalities

J. E. Pin & P. Weil (1996)