Hyperconnections

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What are hyperconnections?

Answer: hyperconnected functors over a fixed base

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "F"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{f}{\Leftarrow}"] & \mathbb{Y} \arrow[dl, "Y"] \\ & \mathbb{B} & \end{tikzcd}$$

morphism of hyperconnections

hyperconnected functors

Recall from topos theory the hypeconnected localic factorization

$$\begin{tikzcd} \mathcal{X} \arrow[r, two heads, "F"] & \cdot \arrow[r] & \mathcal{Y} \end{tikzcd}$$

hyp — localic

of geometric morphisms

Algebraic direction

$$\begin{tikzcd} \mathcal{Y} \arrow[r, tail] & \cdot \arrow[r, two heads] & \mathcal{X} \end{tikzcd}$$

localic — hyp

F hyper connected

$\Updownarrow$

F^* preserves Ω

Algebraic direction

What are hyperconnections for restriction categories?

Restriction category:

$\mathbb{X}$ with "restriction" combinator

$$\frac{A \xrightarrow{f} B}{A \xrightarrow{\overline{f}} B}\ \overline{(\ )}$$

[r.1] $\overline{f}f = f$

[r.2] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$

[r.3] $\overline{f}\,\overline{g} = \overline{\overline{f}g}$

[r.4] $h\overline{f} = \overline{hf}\,h$

algebraic formulation of partial maps

$f \leq g \Leftrightarrow \overline{f}g = f$ restriction order

$\mathcal{O}(A) = \{e : A \to A \mid e = \overline{e}\}$

the open sets of A or the admissible predicates

restriction functors preserve restriction $F(\overline{g}) = \overline{F(g)}$

Defn: $F : \mathbb{X} \longrightarrow \mathbb{Y}$, a restriction functor, is $\boxed{\text{hyperconnected}}$ in case for each $X \in \mathbb{X}$

$$F|_{\mathcal{O}(X)} : \mathcal{O}(X) \longrightarrow \mathcal{O}(F(X))$$

is an isomorphism.

preserves and reflects admissible predicates algebraic direction

The localic / hyperconnected factorization for restriction functors:

Defn: A restriction functor is $\boxed{\text{localic}}$ iff

- it is bijective on objects
- for every map $F(A) \xrightarrow{g} F(B)$ the poset of maps f with $g \leq F(f)$ is non-empty and downward directed.

Restriction cats as enriched cats with Richard Garner

Proposition: The category of restriction categories with restriction functors admits a factorization system of functors into localic followed by hyperconnected functors.

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "F"] \arrow[dr, "L"'] & & \mathbb{Y} \\ & \mathbb{Z} \arrow[ur, "H"'] & \end{tikzcd}$$

localic

hyperconnection

What are hyperconnections for join restriction categories?

Answer: hyperconnected restriction functors

.... which are join restriction functors!

Do they induce a factorization system?

What does it look like?

A harder question to answer concretely!

What are join restriction categories?

Join restriction categories (join categories) are restriction categories with joins of compatible sets of parallel maps.

compatible:

$f, g : A \to B$

$f \smile g \iff \bar{f} g = \bar{g} f$

Compatible set:

$\mathcal{F} \subseteq \mathbb{X}(A, B)$ such that $\forall f, g \in \mathcal{F} .\ f \smile g$

join functors preserve

- restriction $F(\bar{f}) = \overline{F(f)}$
- joins $F(\bigvee \mathcal{F}) = \bigvee F(\mathcal{F})$

join of compatible set:

$\bigvee \mathcal{F} : A \to B$

- $f \in \mathcal{F} \Rightarrow f \leq \bigvee \mathcal{F}$
- $\forall f \in \mathcal{F}, f \leq g \Rightarrow \bigvee \mathcal{F} \leq g$

least upper bound

$\mathcal{O}(A)$ is a locale

with

$h(\bigvee \mathcal{F})k$

$= \bigvee \{ hfk \mid f \in \mathcal{F} \}$

Join categories have <u>intersections</u> of parallel maps

$$f \wedge g = \bigvee \{ h \mid h \leq f, h \leq g \}$$

these are not very well-behaved

$F : \mathbb{X} \to \mathbb{Y}$

<u>Defn</u>: A join restriction functor is <u>localic</u> in case:

- It is bijective on objects
- For every $y : Y_1 \to Y_2 \in \mathbb{Y}$ there is a decomposition $y = \bigvee y_i$ and maps $x_i : F^{-1}(Y_1) \to F^{-1}(Y_2)$ such that $F(x_i) \geq y_i$.

  the x_i need not be compatible
- F preserves intersections

<u>Proposition</u>: Localic join functors are orthogonal to hyper connected join functors

$$\begin{tikzcd}
\mathbb{X} \arrow[r, "L"] \arrow[d, "F"'] & \mathbb{Y} \arrow[d, "G"] \arrow[dl, dashed, "J"'] \\
\mathbb{A} \arrow[r, two heads, tail, "H"'] & \mathbb{B}
\end{tikzcd}$$

localic

hyperconnected

to get a factorization all we need is factorizations.. ..

Hmmm..!

? Now for something completely different ?

An (internal) $\boxed{I\text{-partite category}}$, $\mathbb{B}$, in a join restriction category $\mathbb{X}$ consists of:

- For each $i \in I$ an object $B(i)$
- For each pair $(i,j) \in I \times I$ a span

objects of objects

objects of arrows

$$\begin{tikzcd} & B(i,j) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ B(i) & & B(j) \end{tikzcd}$$

s, t total and where pullbacks along s exist

- For each $i,j,k \in I$ a composition $\mu_{i,j,k}$

$$\begin{tikzcd} & & B(i,j,k) \arrow[rr, "\mu"] \arrow[dl] \arrow[dr] \arrow[d, phantom, "\vee"] & & B(i,k) \arrow[d, "t"] \arrow[ddllll, "s"'] \\ & B(i,j) \arrow[dl, "s"'] \arrow[dr, "t"] & & B(j,k) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ B(i) & & B(j) & & B(k) \end{tikzcd}$$

a $\{*\}$-partite category is an internal category

- For each $i \in I$ a unit $e : B(i) \longrightarrow B(i,i)$

- Satisfying the usual diagrams to make this an associative composition with units.

Externalizing I-partite categories:

Given an (internal) I-partite category ~~[scribbled]~~ $\mathbb{B}$ in a join category $\mathbb{X}$, one can form a hyperconnection over $\mathbb{X}$:

$$\begin{tikzcd} \widehat{\mathbb{B}} \arrow[d, "B"'] \\ \mathbb{X} \end{tikzcd}$$

$\therefore$ $B(a) = at$

$\therefore$ $B(\bar{a}) = B(\bar{a}e) = \cancel{B}\,\bar{a}et = \bar{a}$
so hyperconnected

objects: $i \in I$

maps: $a : i \longrightarrow j$ are partial sections

$$\begin{tikzcd} B(i) \arrow[r, dashed, "a"] & B(i,j) \arrow[l, bend left, "s"] \end{tikzcd} \qquad a s = \bar{a}$$

Composition: $\dfrac{a : i \to j \quad b : j \to k}{\langle a\bar{t}b, atb\rangle \mu : i \longrightarrow j}$

$\therefore$

$$\begin{tikzcd} B_i \arrow[r, dashed, "\langle a\bar{t}b,\, atb\rangle"] \arrow[dr, bend right, "a\bar{t}b"'] \arrow[drrr, bend left, "atb"] & B_{ijk} \arrow[rr, "\mu_{ijk}"] \arrow[d] \arrow[drr] & & B_{ik} \\ & B_{ij} \arrow[dr, "t"'] & & B_{jk} \arrow[dl, "s"] \\ & & B_j & \end{tikzcd}$$

identity: $e : B(i) \longrightarrow B(i,i)$

restriction: $\dfrac{a : i \to j}{\bar{a}e : i \to i}$ join: $\vee a_i$

Internalizing join functors over a base:

Let $\mathbb{X} \xrightarrow[F]{} \mathbb{B}$ be a join functor where we now assume $\mathbb{B}$ has glueings or is manifold complete. We construct a $\mathbb{X}_0$-partite category $\widetilde{F}$:

Manifold construction due to Grandis

$$\widetilde{F}(x) = F(x)$$

$\widetilde{F}(x,y)$ is given by the atlas

pullbacks of étale maps exist!

- charts: for each $f \in \mathbb{X}(x,y)$ a copy of $F(x)$
- glueings: for each $f, g \in \mathbb{X}(x,y)$

$$\overline{F(f \wedge g)} : F(x)_f \longrightarrow F(x)_g$$

a local atlas

This is an étale map or local homeomorphism

$F(x)_f \xrightarrow{1} F(x)$

$f: F(x)_f \longrightarrow F(y)$

$$\begin{tikzcd} & \widetilde{F}(x,y) \arrow[dl, "s"'] \arrow[dr, "t"] & \\ F(x) & & F(y) \end{tikzcd}$$

This gives a source étale $\mathbb{X}_0$-partite category

$\boxed{\text{Theorem :}}$ There is a Galois adjunction between join functors over $\mathbb{B}$ (with gluings) and partite categories with cofunctors within $\mathbb{B}$

$$\begin{tikzcd} Jn(\mathbb{B}) \arrow[r, bend left, "In"] \arrow[r, phantom, "\perp"] & Part(\mathbb{B}) \arrow[l, bend left, "Ex"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{X} \arrow[rr, "F"] \arrow[dr, "x"'] & \arrow[d, phantom, "\Leftarrow"] & \mathbb{Y} \arrow[dl, "y"] \\ & \mathbb{B} & \end{tikzcd} \qquad \begin{tikzcd} \hat{\mathbb{X}} \arrow[r, squiggly, "\hat{F}"] & \hat{\mathbb{Y}} \end{tikzcd}$$

!! Amazingly !! the unit is a localic functor

$$\begin{tikzcd} \mathbb{X} \arrow[r, "\eta"] \arrow[d, "F"'] \arrow[dr, phantom, "="] & \tilde{\hat{F}} \arrow[dl, "U_{\hat{F}}"] \\ \mathbb{B} & \end{tikzcd}$$

... which gives the factorization we sought!

$$\begin{tikzcd} \mathbb{X} \arrow[r, "F"] \arrow[d, "\eta"'] & \mathbb{Y} \arrow[r, "\eta"] & Man(\mathbb{Y}) \\ \widehat{Fi} \arrow[ur, dashed] \arrow[urr, "U_{\widehat{Fi}}"'] & & \end{tikzcd}$$

localic

hyperconnected

Examples :

- $\underline{Par}$ (sets and partial maps) has every map etale .... so an internal category gives an external join monoid sitting over Par by a hyperconnection

just a small category!

$$\begin{tikzcd} \underline{M} \arrow[d] \\ \underline{Par} \end{tikzcd}$$

$\mathcal{O}(*)$ is a complete atomic Boolean algebra

Special example (due to [Lawson])

$$P_n = \langle a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1} : a_i^{-1}a_i = 1, \; a_i^{-1}a_j = 0, \; i \neq j \rangle$$

polycyclic monoid

inverse monoid with 0

$$\begin{tikzcd} P_n \arrow[d, "Ex"] \\ C_n \arrow[d, phantom, "\cup"] \\ \text{Total isos } V_{n,1} \end{tikzcd}$$

C_n: Cuntz étale monoid

Total isos $V_{n,1}$: the Thompson group

- $Id : \mathrm{Par} \longrightarrow \mathrm{Par}$ is a join functor

$$\widehat{Id}(X, Y) = X \times Y$$

More generally for any étale join category (all maps local homeo) with gluings

$$\widehat{Id}(X, Y) = X \times Y$$

the product in the category of local homeomorphisms [Selinger]

- $\underline{\mathrm{Scheme}} \xrightarrow{Id} \underline{\mathrm{Scheme}}$

For any R,

$$\begin{array}{ccc} & \widehat{Id}(R, \mathbb{Z}[x]) & \\ {}^{s}\swarrow & & \searrow \\ R & & \mathbb{Z}[x] \end{array}$$

the structure sheaf of R

- The Haefliger groupoid

classifying foliations?

$$\mathrm{ParDiffeo}(\mathbb{R}^n) \xrightarrow{i} \mathrm{OpenPar}(\mathrm{Man}) \qquad \mathrm{Int}(i)$$

internal étale groupoid

partial diffeomorphisms $\mathbb{R}^n \longrightarrow \mathbb{R}^n$

Manifolds with open partial differentiable maps

- The "full group" or "full topological group"

$$G \xrightarrow{A} \underline{\mathrm{Set}} \qquad\qquad G \xrightarrow{A} \mathrm{Top}_{open}$$

Group

action of group on set

action of group on topological space

$$\mathrm{Grpd}(\mathrm{Ex}(\mathrm{Int}(A)))$$

full ...

- Join categories are internal partite categories in Par(Locale) ...

Given a join category, $\mathcal{X}$, there is a fundamental functor

$$\mathcal{O} : \mathcal{X} \longrightarrow \mathrm{Par}(\mathrm{Locale})\ ; \quad \begin{array}{c} X \\ \downarrow f \\ Y \end{array} \longmapsto \begin{array}{c} \mathcal{O}(X) \\ \updownarrow f^* \\ \mathcal{O}(Y) \end{array} \quad \begin{array}{c} Fe \\ \uparrow \\ e \end{array}$$

this is always a hyperconnection ....

SO a join inverse category corresponds precisely to a source étale internal partite category in Par(Locale).