

Hyperconnections

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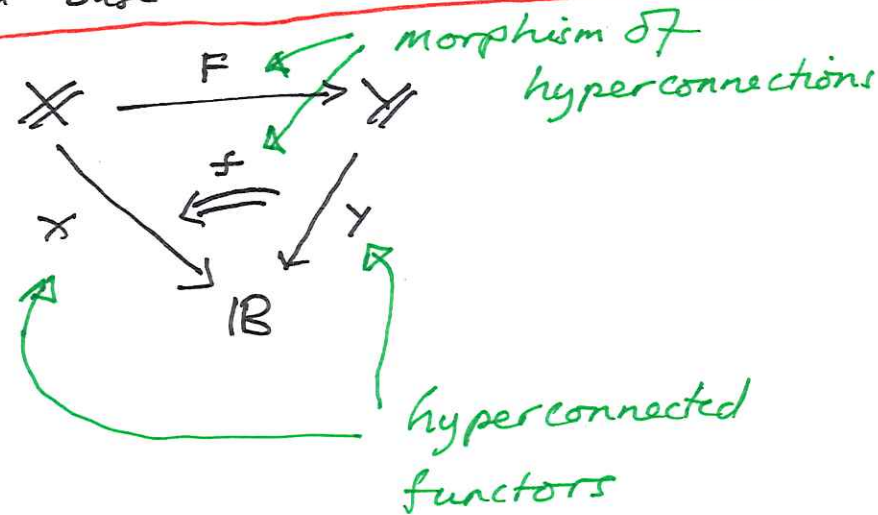
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Work with Richard Garner

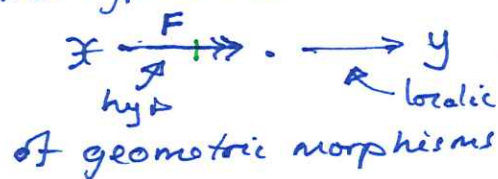
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What are hyperconnections?

Answer: hyperconnected functors over a fixed base



Recall from topos theory
the hyperconnected localic factorization



F hyper connected
 $\updownarrow$
 F^* preserves Ω

Algebraic direction

What are hyperconnections for restriction categories?

Restriction category:

$\mathbb{X}$ with "restriction" combinator

$$\frac{A \xrightarrow{f} B}{A \xrightarrow{\bar{f}} B} (\bar{})$$

[r.1] $\bar{f}f = f$

[r.2] $\bar{f}\bar{g} = \bar{g}\bar{f}$

[r.3] $\bar{f}\bar{g} = \overline{fg}$

[r.4] $h\bar{f} = \overline{hf}h$

algebraic formulation of partial maps

$$f \leq g \Leftrightarrow \bar{f}g = f$$

restriction order

$$\mathcal{O}(A) = \{e: A \rightarrow A \mid e = \bar{e}\}$$

the open sets of A or the admissible predicates

restriction functors preserve restriction
 $F(\bar{g}) = \overline{F(g)}$

Defn: $F: \mathbb{X} \rightarrow \mathbb{Y}$, a restriction functor, is hyperconnected in case for each $x \in \mathbb{X}$

$F|_{\mathcal{O}(x)}: \mathcal{O}(x) \rightarrow \mathcal{O}(F(x))$ is an isomorphism.

preserves and reflects admissible predicates

Algebraic direction

The localic / hyperconnected factorization for restriction functors:

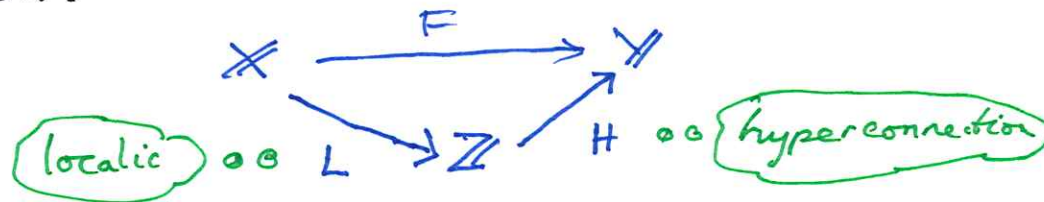
Defn: A restriction functor is localic iff

- it is bijective on objects
- for every map $F(A) \xrightarrow{g} F(B)$ the poset of maps f with $g \leq F(f)$ is non-empty and downward directed.

Proposition: The category of restriction categories

Restriction cats
as enriched cats
with Richard
Garner

with restriction functors admits a factorization system of functors into localic followed by hyperconnected functors.



What are hyperconnections for join restriction categories?

Answer: hyperconnected restriction functors
.... which are join restriction functors!

Do they induce a factorization system?

What does it look like?

A harder question
to answer
concretely!

What are join restriction categories?

Join restriction categories (join categories) are restriction categories with joins of compatible sets of parallel maps.

Join functors preserve

- restriction $F(f) = \overline{F(f)}$
- joins $F(V\mathcal{F}) = V F(\mathcal{F})$

Compatible:

$$f, g : A \rightarrow B$$

$$f \cup g \Leftrightarrow \bar{f}g = \bar{g}f$$

Compatible set:

$$\mathcal{F} \subseteq X(A, B)$$

such that $\forall f, g \in \mathcal{F}. f \cup g$

Join of compatible set:

$$V\mathcal{F} : A \rightarrow B$$

- $f \in \mathcal{F} \Rightarrow f \leq V\mathcal{F}$
- $\forall f \in \mathcal{F}, f \leq g \Rightarrow V\mathcal{F} \leq g$

least upper bound

$\mathcal{O}(A)$ is a locale

with

$$h(V\mathcal{F})k$$

$$= V\{hfk \mid f \in \mathcal{F}\}$$

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Join categories have intersections of parallel maps

$$f \wedge g = \bigvee \{ h \mid h \leq f, h \leq g \}$$

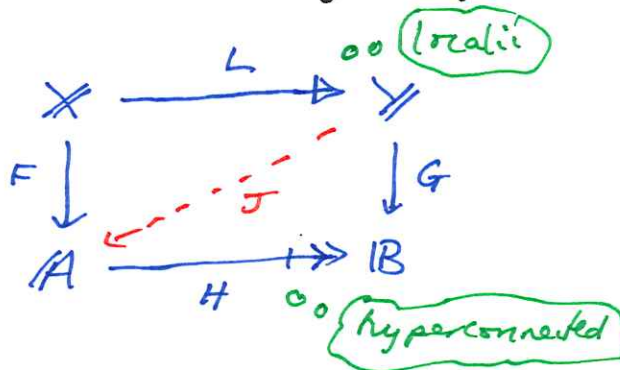
these are not very well-behaved

$$F: \mathbb{X} \rightarrow \mathbb{Y}$$

Defn: A join restriction functor is localic in case:

- It is bijective on objects
 - For every $y: Y_1 \rightarrow Y_2 \in \mathbb{Y}$ there is a decomposition $y = \bigvee y_i$ and maps $x_i: F^{-1}(Y_1) \rightarrow F^{-1}(Y_2)$ such that $F(x_i) \geq y_i$.
 - F preserves intersections
- the x_i need not be compatible

Proposition: Localic join functors are orthogonal to hyper connected join functors



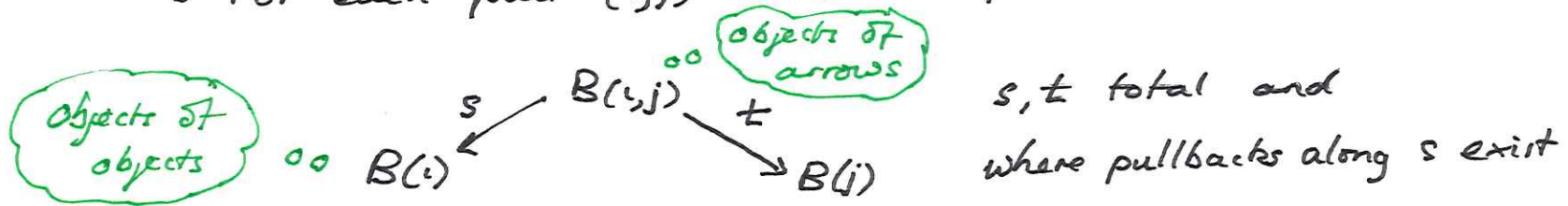
to get a factorization all we need is factorizations...

Hmmm...!

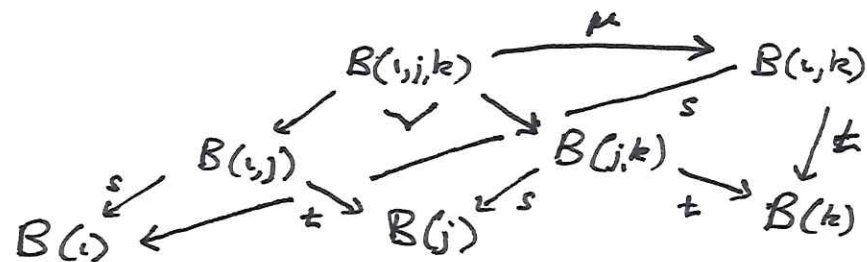
? Now for something completely different?

An (internal) I -partite category, $\mathcal{B}$, in a join restriction category $\mathcal{X}$ consists of:

- For each $i \in I$ an object $B(i)$
- For each pair $(i, j) \in I \times I$ a span



- For each $i, j, k \in I$ a composition $\mu_{i, j, k}$



a $\{*\}$ -partite category is an internal category

- For each $i \in I$ a unit $e: B(i) \rightarrow B(i, i)$
- Satisfying the usual diagrams to make this an associative composition with units.

Externalizing I-partite categories:

Given an (internal) I-partite category B in a join category $\mathbb{X}$, one can form a hyperconnection over $\mathbb{X}$:

$$\begin{array}{c} \widehat{B} \\ \downarrow B \\ \mathbb{X} \end{array}$$

$$B(a) = at$$

$$B(\bar{a}) = B(\bar{a}e) = \bar{a}et = \bar{a} \quad \text{so hyperconnected}$$

objects: $i \in I$

maps: $a: i \rightarrow j$ are partial sections

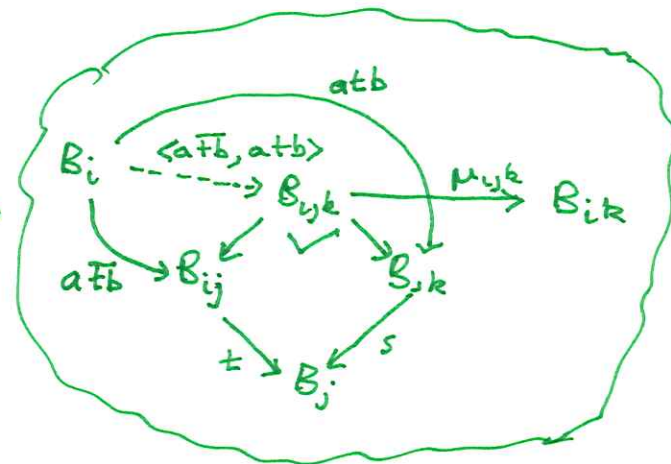
$$\begin{array}{c} a \dashrightarrow B(i,j) \\ \swarrow s \\ B(i) \end{array}$$

$$as = \bar{a}$$

$$\text{Composition: } \frac{a: i \rightarrow j \quad b: j \rightarrow k}{\langle a\bar{t}b, atb \rangle \mu: i \rightarrow k}$$

$$\text{Identity: } e: B(i) \rightarrow B(i,i)$$

$$\text{restriction: } \frac{a: i \rightarrow j}{\bar{a}e: i \rightarrow i} \quad \text{join: } \forall a_i$$



Internalizing join functors over a base:

Let $\mathbb{X} \xrightarrow{F} B$ be a join functor where we now assume B has glueings or is manifold complete. We construct a $\mathbb{X}_0$ -partite category $\tilde{F}$:

oo Manifold construction due to Grandis

$$\tilde{F}(x) = F(x)$$

$\tilde{F}(x, y)$ is given by the atlas

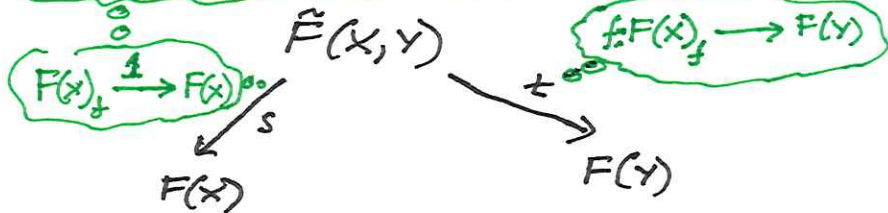
pullbacks of étale maps exist!

- charts: for each $f \in \mathbb{X}(x, y)$ a copy of $F(x)$
- glueings: for each $f, g \in \mathbb{X}(x, y)$

oo a local atlas

$$\overline{f \wedge g} : F(x)_f \rightarrow F(x)_g$$

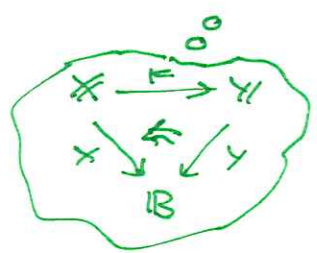
This is an étale map or local homeomorphism



This gives a source étale $\mathbb{X}_0$ -partite category

Theorem : There is a Galois adjunction between
 join functors over $\mathcal{B}$ (with glueings) and
 partite categories with cofunctors within $\mathcal{B}$

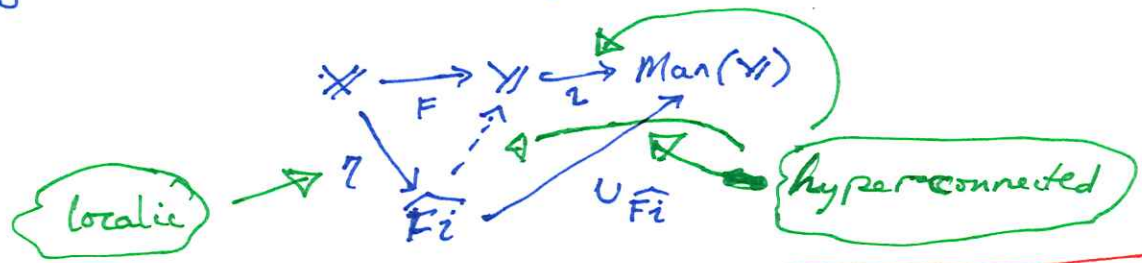
$$J_n(\mathcal{B}) \begin{matrix} \xrightarrow{In} \\ \perp \\ \xleftarrow{Ex} \end{matrix} Part(\mathcal{B})$$



!! Amazingly !! the unit is a localic functor

$$\begin{array}{ccc} X & \xrightarrow{p} & \widehat{F} \\ F \downarrow & \cong & \swarrow U_F \\ \mathcal{B} & & \end{array}$$

... which gives the factorization we sought!



Examples:

- Par (sets and partial maps) has every map étale.... so an internal category gives an external join monoid sitting over Par by a hyperconnection

$\underline{M}$
↓
Par

just a small category!

$\mathcal{C}(\ast)$ is a complete atomic Boolean algebra

Special example (due to Lawson)

$$P_n = \langle a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1} : a_i^{-1} a_i = 1, a_i^{-1} a_j = 0, i \neq j \rangle$$

inverse monoid with 0

polycyclic monoid

↓ Ex

C_n

Cuntz étale monoid

⋔

Total isos
 $V_{n,1}$

the Thompson group

- $\text{Id} : \text{Par} \rightarrow \text{Par}$ is a join functor

$$\widehat{\text{Id}}(X, Y) = X \times Y$$

More generally, for any étale join category
(all maps local homes) with glueings

$$\widehat{\text{Id}}(X, Y) = X \times Y \quad \bullet \bullet$$

the product in the
category of local
homeomorphisms
[Selinger]

- $\underline{\text{Scheme}} \xrightarrow{\text{Id}} \underline{\text{Scheme}}$

For any R ,

$$\begin{array}{ccc} & \widehat{\text{Id}}(R, \mathbb{Z}[x]) & \\ \swarrow s & & \searrow \\ R & & \mathbb{Z}[x] \end{array}$$

the structure sheaf
of R

- The Haefliger groupoid

classifying
foliations?

internal
étale
groupoid

$$\text{ParDiffeo}(\mathbb{R}^n) \xrightarrow{i} \text{OpenPar}(\text{Man})$$

$\text{Int}(i)$

partial diffeomorphisms
 $\mathbb{R}^n \rightarrow \mathbb{R}^n$

Manifolds with
open partial
differentiable maps

- The "full group" or "full topological group"

$$G \xrightarrow{A} \underline{\text{Set}}$$

Group

action of
group on set

$$G \xrightarrow{A} \text{Topopen}$$

action of
group on
topological space

$$\text{Grpd}(\text{Ex}(\text{Int}(A))) \approx \text{full} \dots$$

- Join categories are internal partite categories in $\text{Par}(\text{Locale})$...

Given a join category, $\mathbb{X}$, there is a fundamental functor

$$\Theta: \mathbb{X} \longrightarrow \text{Par}(\text{Locale}); \quad \begin{array}{c} \mathbb{X} \\ \downarrow f \\ Y \end{array} \longmapsto \begin{array}{ccc} \Theta(X) & & Fe \\ \uparrow f^* & & \uparrow \\ \Theta(Y) & & e \end{array}$$

this is always a hyperconnection

SO a join inverse category corresponds precisely to a source étale internal partite category in $\text{Par}(\text{Locale})$.