FAST-GROWING CLONES

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ABSTRACT CLONES

the algebraic notion of Lawvere theory.

Def: $\mathcal{C} = \{ C_n \in \underline{\underline{Set}} \}_{n \in \mathbb{N}}$

$\mathcal{V} = \{ v_i^{(n)} \in C_n \}_{0 \leq i < n \in \mathbb{N}}$

$\mathcal{S} = \{ S_{m,n} : C_m \times C_n^{\,m} \longrightarrow C_n \}_{m,n \in \mathbb{N}}$

subject to axioms.

CONCRETE CLONES

($\mathscr{C}$ with finite powers)

Def $\mathscr{C}\langle X\rangle = \{ \mathscr{C}(X^n, X) \}_{n\in\mathbb{N}}$

$\pi = \{ \pi_i^{(n)} \in \mathscr{C}(X^n, X) \}_{0\le i< n\in\mathbb{N}}$

$$\{ \mathscr{C}(X^m, X) \times \mathscr{C}(X^n, X)^m \cong \mathscr{C}(X^m, X) \times \mathscr{C}(X^n, X^m) \longrightarrow \mathscr{C}(X^n, X) \}_{m,n\in\mathbb{N}}$$

CONCRETE CLONES

($\mathcal{C}$ with finite powers)

Def

$$\mathcal{C}\langle X\rangle = \{\, \mathcal{C}(X^n, X) \,\}_{n\in\mathbb{N}}$$

$$\pi = \{\, \pi_i^{(n)} \in \mathcal{C}(X^n, X) \,\}_{0\le i<n\in\mathbb{N}}$$

$$\Big\{ \mathcal{C}(X^m, X) \times \mathcal{C}(X^n, X)^m \cong \mathcal{C}(X^m, X) \times \mathcal{C}(X^n, X^m) \longrightarrow \mathcal{C}(X^n, X) \Big\}_{m,n\in\mathbb{N}}$$

Universal Algebra Folklore

Every abstract clone can be represented by a concrete clone on a set.

ENRICHED CLONES

Def: [Borceux & Day enriched theories]

Discrete finitary enriched abstract clones in a monoidal category with finite powers $\mathcal{S}$

$$C = \{ C_n \in \mathcal{S} \}_{n \in \mathbb{N}}$$

$$v = \{ \langle v_i^{(n)} \rangle_i : I \longrightarrow C_n^{\,n} \text{ in } \mathcal{S} \}_{n \in \mathbb{N}}$$

$$s = \{ s_{m,n} : C_m \otimes C_n^{\,m} \longrightarrow C_n \text{ in } \mathcal{S} \}_{m,n \in \mathbb{N}}$$

satisfying axioms.

Def: [Power's discrete countable enriched Lawvere theories]

Discrete countable enriched abstract clones in a monoidal category with countable powers $\mathcal{S}$

$$C = \{ C_n \in \mathcal{S} \}_{n \in \overline{N}}$$

$$v = \{ \langle v_i^{(n)} \rangle_i : I \longrightarrow C_n{}^n \text{ in } \mathcal{S} \}_{n \in \overline{N}}$$

$$s = \{ s_{m,n} : C_m \otimes C_n{}^m \longrightarrow C_n \text{ in } \mathcal{S} \}_{m,n \in \overline{N}}$$

satisfying axioms.

$$\overline{N} = N \cup \{\omega\}$$

[Tbilisi Math. J. 10(3): 297–328, 2017]

ENRICHED EMBEDDING

- Countable case

  $\mathcal{S}$ a monoidal closed category with countable powers

$$\frac{C_n \otimes C_\omega^{\,n} \xrightarrow{s_{n,\omega}} C_\omega}{C_n \longrightarrow \underline{\mathcal{S}}(C_\omega^{\,n}, C_\omega)}$$

provides a positive embedding

[Tbilisi Math. J. 10(3): 297–328, 2017]

ENRICHED EMBEDDING AND REPRESENTATION

- Countable case

$\mathcal{S}$ a monoidal closed category with countable powers

$$\dfrac{C_n \otimes C_\omega^{\,n} \xrightarrow{\;s_{n,\omega}\;} C_\omega}{C_n \longrightarrow \underline{\mathcal{S}}(C_\omega^{\,n}, C_\omega)}$$

provides a positive embedding that in the presence of equalisers restricts to a positive representation.

[Tbilisi Math. J. 10(3): 297–328, 2017]

ENRICHED EMBEDDING AND REPRESENTATION

- Finitary case

  $\mathcal{S}$ a monoidal biclosed category with countable powers, equalisers, and colimits of ω-chains of sections preserved by finite powers

[Tbilisi Math. J. 10(3):297–328, 2017]

ENRICHED EMBEDDING AND REPRESENTATION

- Finitary case

$\mathcal{S}$ a monoidal biclosed category with countable powers, equalisers, and <u>colimits of ω-chains of sections preserved by finite powers</u>

$$\begin{tikzcd} \underline{\mathcal{C}\ell}_\omega(\mathcal{S}) \arrow[d, shift left=2] \arrow[d, phantom, "\dashv"] \\ \underline{\mathcal{C}\ell}(\mathcal{S}) \arrow[u, bend left, shift left=2] \end{tikzcd} \qquad \begin{tikzcd} C^\# \\ C \arrow[u, mapsto] \end{tikzcd}$$

$C^\#_\omega = \operatorname{colim}_n \; C_0 \to C_1 \to \cdots C_n \to \cdots$

positively represents C.

QUESTIONS

- Are there embedding and representation theorems for general enriched abstract clones?

- Is infinitary structure necessary for embedding/representation theorems?

  In particular, what about clone embeddings in the topos of finite sets?

CLONES IN FINITE SETS

- Abstract clones are monads.
- Concrete clones are double-dualisation (or continuation) monads

$$K_R(X) = (X \Rightarrow R) \Rightarrow R$$

CLONES IN FINITE SETS

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$$K_R(X) = (X \Rightarrow R) \Rightarrow R$$

- Are there fast-growing clones?

clones C such that for all $r \in \mathbb{N}$,
there is $n \in \mathbb{N}$ with $C_n > r^{(r^n)}$

THE SELECTION MONAD

[Escardó & Oliva]

$$\begin{tikzcd} \mathbb{J}_R\mathcal{C} \arrow[r, bend left] \arrow[r, phantom, "\lambda"] & \mathcal{C} \arrow[l, bend left] \end{tikzcd}$$

$$J_R(X) = (X \Rightarrow R) \Rightarrow X$$

$$\mathbb{J}_R\mathcal{C}(A, B) = \mathcal{C}(R^B \times A, B)$$

THE SELECTION MONAD

[Escardó & Oliva]

($\mathcal{C}$ a CCC)

$$
\begin{tikzcd}
\mathcal{J}_R\mathcal{C} \arrow[rr] \arrow[dr, bend left=15] & & \mathcal{K}_R\mathcal{C} \arrow[dl, bend left=15] \\
& \mathcal{C} \arrow[ul, bend left=40] \arrow[ur, bend right=40] \arrow[loop left] \arrow[loop right] &
\end{tikzcd}
$$

$J_R(X) = (X \Rightarrow R) \Rightarrow X$ $\qquad$ $K_R(X) = (X \Rightarrow R) \Rightarrow R$

$$\mathcal{J}_R\mathcal{C}(A,B) = \mathcal{C}(R^B \times A, B) \longrightarrow \mathcal{C}(R^B, R^A) = \mathcal{K}_R\mathcal{C}(A,B)$$

FAST-GROWING CLONES

Lemma There is a distributive law

$$T J_R \Rightarrow J_R T$$

for every strong monad T.

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for every strong monad T.

Thm The monad $J_2 K_2$ on finite sets is fast growing. Hence, there is no embedding theorem in the topos of finite sets.

Cor. There are monads on the topos of finite sets with free algebras that asymptotically grow faster than every iterated exponential, for any natural height, on their generators

<u>Cor</u>. There are monads on the topos of finite sets with free algebras that asymptotically grow faster than every iterated exponential, for any natural height, on their generators.

QUESTIONS

- Algebraic/combinatorial constructions of fast-growing finite clones?
- Study of algebraic theories on finite sets.