Compositional Economic Game Theory

Neil Ghani

and Julian Hedges, Viktor Winschel, Philipp Zahn,
MSP group, The Scottish Free State

- **Compositionality:** Operators build big games from small games
  - Lifting results about parts of a game to the whole game.
  - Crucial to understand: this is bottom up, not top down.
  - Optimal strategies for compound games from optimal strategies of their subcomponents!

- **Motivation:** Software $\Leftarrow$ Compositionality $\Leftarrow$ Structure $\Leftarrow$ Category Theory
  - Difficult $\Rightarrow$ new concepts, eg *coutility*, *utility-indexed games*
  - **You can learn economic game theory by learning category theory, the modelling language of the future**

Structure

- **Part 1:** Good news, compositionality seems possible

- **Part 2:** Bad news, developing a theory becomes painful to the point of crucifixtion.

- **Part 3:** Resurrection! Category theory saves the day!!!!

Part I: Simple Games (Apologies from a Non-Expert to Experts!)

- **Defn:** A *basic game* consists of
  - A set of actions A the player can take, and a set U of utilities
  - A function $f : A \to U$ assigning to each action, a *utility*

- **Defn:** Optimal actions/equilibria for a simple game are

$$\mathsf{Eq}(A, U, f) = \mathsf{argmax}\ f = \{a \in A \mid (\forall a' \in A) fa \geq fa'\}$$

- **Question:** Is this definition correct for a two player game?

$$f : A_1 \times A_2 \to U_1 \times U_2$$

- **Motivation:** Two prisoners face a choice
  - Each is under pressure to report criminal behaviour of the other to the authorities.
  - They can cooperate with each other, or defect $\Rightarrow A = \{C, D\}$
  - Utilities are given by $f : A \times A \to Z \times Z$

$$\begin{array}{rr} f(C,C) = (0,0) & f(D,C) = (1,-3) \\ f(C,D) = (-3,1) & f(D,D) = (-2,-2) \end{array}$$

- **Conclusion:** The best strategy for each player is to defect!
  - Rather depressing for utopians! Assumptions: no communication, no future cost for bad behaviour etc.

No! Example = Nash Equilibria

- **Motivation:** Simple game equilibria doesn't compute the optimal strategy in the prisoner's dilemma

- **Defn:** A 2-player game is
  - Sets of actions A_1, A_2 and utilities U_1, U_2 of utilities
  - A function $f : A_1 \times A_2 \to U_1 \times U_2$ assigning to each pair of actions, a pair of utilities

- **Defn:** Optimal actions/equilibria for a 2-player game are given by $\mathsf{Nash} \subseteq A_1 \times A_2$

$$
\begin{aligned}
(a_1, a_2) \in \mathsf{Nash}\ f \quad \text{iff} \quad & a_1 \in \mathsf{argmax}\ (\pi_1 \circ f(-, a_2)) \\
& \wedge a_2 \in \mathsf{argmax}\ (\pi_2 \circ f(a_1, -))
\end{aligned}
$$

- **Key Idea:** Nash equilibria are given as primitive.
  - This is not a compositional definition as the definition is not derived from equilibria for simpler games
  - It is simply postulated as reasonable, justified empirically.

- **Question:** Is there no operator which combines two 1-player games into a 2-player game?
  - And defines the equilibria of the derived game via those of the component games.

- **Remark:** Of course this is difficult as optimal moves for one game may not remain optimal when that game is incorporated into a networked collection of games.

From Games to Utility Free Games

- **Defn:** A *utility-free game* consists of
  - A set A of moves, a set U of utilities and an equilibria function $\mathsf{E} : (A \to U) \to \mathsf{P}A$ where P is powerset
  - The set of utility-free games with actions A and utilities U is written $\mathsf{UF}_A U$

- **Key Idea:** These games leave the utility function abstract
  - The equilibria is given for *every* potential utility function
  - And its not always argmax, eg Nash

- **Defn:** Let $G_1 \in \mathsf{UF}_{A_1}U_1$ and $G_2 \in \mathsf{UF}_{A_2}U_2$ be UF-games. Their monoidal product is the UF-game

$$G_1 \otimes G_2 : \mathsf{UF}_{A_1 \times A_2}(U_1 \times U_2)$$

with equilibrium function

$$\begin{aligned}(a_1, a_2) \in \mathsf{E}_{G_1 \otimes G_2} k \quad \text{iff} \quad & a_1 \in \mathsf{E}_{G_1}(\pi_1 \circ k(-, a_2)) \quad \wedge \\ & a_2 \in \mathsf{E}_{G_2}(\pi_2 \circ k(a_1, -))\end{aligned}$$

- **Thm:** The above looks like Nash. Indeed, we have a beautiful equation ....

$$\mathsf{Nash} = \mathsf{argmax} \otimes \mathsf{argmax}$$

- **Key Idea:** CGT is possible. Don't hardwire a specific utility.

Part II: Our Idea ..... Open Games

- **Motivation:** Simple games possess limited structure, and hence support limited operators
  - More operators $\Rightarrow$ more compositionality
  - Lets develop a more complex model!
- **Example:** Lets place a bet
  - I have a bank balance. I have different strategies. These factors decide on my bet which I give to the bookmaker
  - The bookmaker has a variety of strategies to deal with my bet. When the event is finished, he returns my winnings
  - A forwards world of physical action, a backwards world of reflection on possible consequences of action.

Coutility needed for Conservation of Utility

- **Types:** Let X, Y, S, R be sets. Think of X as the game's state.
  - Y is move or other observable action
  - R is utility which the environment produces from a move
  - S is coutility which the system feeds into the environment
- **Examples:** X is my bank balance, the bet that the bookie must react to. External factors affecting our decisions
  - Y is my bet or the action the bookie takes
  - R is my winnings or the utility gained from the move
  - S is the coutility fed back into the system, eg the bookie sends me my winnings.

- **Defn** An open game $G : (X, S) \to (Y, R)$ is defined by
  - A set Σ of strategies
  - A play function $P : \Sigma \times X \to Y$
  - A coutility function $C : \Sigma \times X \times R \to S$
  - An equilibrium function $E : X \times (Y \to R) \to \mathsf{P}\Sigma$

  where P is powerset.

- **Example:** Prisoners Dilemma PD : $(1, 1) \to (M, Z \times Z)$ and strategies M, where $M = \{C, D\}^2$
  - Two round PD: strategies $M \times (M \to M)$, moves M^2, utility $(Z \times Z)^2$

Parallel composition of Open Games (eg, PD from Argmax)

- **Assume:** Given open games

$$G : (X, S) \to (Y, R) \quad \text{and} \quad G' : (X', S') \to (Y', R')$$

- **Define:** Construct an open game

$$G \otimes G' : (X \times X', S \times S') \to (Y \times Y', R \times R')$$

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where $\Sigma_{G \otimes G'} = \Sigma_G \times \Sigma_{G'}$ and

$$P_{G \otimes G'} \; (\sigma, \sigma') \; (x, x') \; = \; (P_G \; \sigma \; x, P_{G'} \; \sigma' \; x')$$

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- **Obs:** Still no category theory, but maybe no need either!

Sequential Composition of Open Games (eg 2 Round Games)

- **Sequential Composition:** Given open games

$$G : (X, S) \to (Y, R) \quad \text{and} \quad H : (Y, R) \to (Z, T)$$

construct an open game

$$H \circ G : (X, S) \to (Z, T)$$

where $\Sigma_{H \circ G} = \Sigma_H \times \Sigma_G$

- **Key Idea:** Note, without coutility we could not formalise how later games create the utility of earlier games.

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C_{H \circ G}\ (\sigma, \sigma')\ x\ t & = & C_G\ \sigma\ x\ (C_H\, \sigma'\, (P_G \sigma x)\, t) \\
(\sigma, \sigma') \in E_{H \circ G}\, x\, (k : Z \to T) & \text{iff} & \sigma \in E_G\, x\, (y \mapsto C_H\ \sigma'\ y\ (k(P_H \sigma' y))) \\
& \wedge & \sigma' \ \in \ E_H\ (P_G \sigma x)\ k
\end{array}$$

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Bring on the Category Theory!

- **What was Good For You?** Some things (hopefully)
  - You learned a little economic game theory
  - You learned that despite the implausibility of its existence, compositional game theory is possible
  - You learned this is non-trivial, eg new concepts needed and games/equilibria must be indexed by all possible utilities
- **What was Bad For You?:** If you are anything like me
  - I distrust random sequences of symbols. My eyes glaze over
  - Were these definitions correct or canonical
  - These definitions are not tractable, eg associativity

- **Definition:** A lens $(X, S) \to (Y, R)$ consists of two functions

$$P : X \to Y \text{ and } C : X \times R \to S$$

- **Observations:** Some simple points
  - Objects which are pairs of sets and maps which are lenses forms a category Lens
  - A map $(1, 1) \to (X, S)$ is just an element of X
  - A map $(Y, R) \to (1, 1)$ is just a function $Y \to R$
  - A game $G : (X, S) \to (Y, R)$ is a Σ-indexed family of lenses $G_\sigma : (X, S) \to (Y, R)$ together with, for each $\sigma \in \Sigma$ a subset $E_\sigma \subseteq \mathsf{Lens}(1, 1)(X, S) \times \mathsf{Lens}(Y, R)(1, 1)$

Composition of Games, via the Composition of Lenses

- **Assume** Given a game $G : \Sigma \to \mathsf{Lens}(X,S)(Y,R)$ with equilibria E_G and one $H : \Sigma' \to \mathsf{Lens}(Y,R)(Z,T)$ with equilibria E_H.

- **Define:** A family of lenses $H \circ G : \Sigma \times \Sigma' \to \mathsf{Lens}(X,S)(Z,T)$ by

$$(H \circ G)(\sigma, \sigma') = (H\sigma') \circ (G\sigma)$$

- **Define:** ... and an equilibrium predicate

$$\begin{array}{rll} (x,k) \in E_{H\circ G}(\sigma,\sigma') & \text{iff} & (x, k \circ H\sigma') \in E_G\sigma \\ & \wedge & (G\sigma \circ x, k) \in E_H\sigma' \end{array}$$

- **Comment:** Blew my mind away, and associativity trivial!

- **Motivation:** We have a monoidal category with 1-cells being games. Lots of string diagrams etc. But, to define games via universal properties, we need maps between games.

- **Assume:** Given a game $G : \Sigma \to \mathsf{Lens}(X, S)(Y, R)$ with equilibria E_G and one $H : \Sigma' \to \mathsf{Lens}(X', S')(Y', R')$ with equilibria E_H.

- **Define** A map $G \to H$ is i) a map of indexes $f : \Sigma \to \Sigma'$; and ii) lenses $\alpha : (X, S) \to (X', S')$ and $\beta : (Y, R) \to (Y', R')$ such that

  - $(\sigma \in \Sigma)\ \ \beta \circ G\sigma = H(f\sigma) \circ \alpha$

  - $(\sigma \in \Sigma)(x : X)(k : Y' \to R')$

  $$(x, k \circ \beta) \in E_G\sigma \Rightarrow (\alpha \circ x, k) \in E_H(f\sigma)$$

- **Comment:** Clinical, clean, powerful and yet tractable.

Summary

- **The Holy Spirit .... :** What we have seen is an example of
  - Category Theory is the heart of Structure
  - Structure and the heart of Compositionality
  - Compositionality is how we understand the world

- **.... Made Flesh:** In our example
  - We developed compositional game theory
  - Highly implausible and rather difficult
  - And impossible without category theory to tame the complexity of computation and an aesthetic to aid discovery

Conclusions:

- **Extensions:** We have also tackled
  - Infinitely Repeated Games via Final Coalgebras
  - Subgame perfection via a categorical modality
  - Mixed Strategies ... next week at ACT

- **Next:** Much more to do
  - More operators, more algorithms
  - Translate into better software
  - Please come and visit or join us at Strathclyde ... send me your CVs!