Segal-type models of weak n-categories

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Strict versus weak n-categories

Idea of strict n-category: in a strict n-category there are cells in dimension $0, \ldots, n$, identity cells and compositions which are associative and unital. Each k-cell has source and target which are $(k-1)$-cells, $1 \leq k \leq n$.

Idea of weak n-category: in a weak n-category there are cells in dimension $0, \ldots, n$, identity cells and compositions which are associative and unital up to an invertible cell in the next dimension, in a coherent way.

An environment for higher categories

To build a model of weak n-category we need a combinatorial machinery that allows to encode:

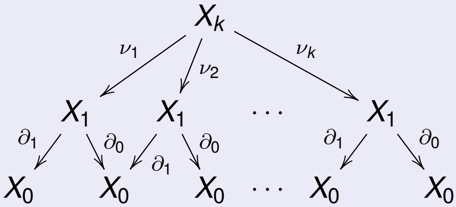
i) The sets of cells in dimension 0 up to n.
ii) The behavior of the compositions (including their coherence laws).
iii) The higher categorical equivalences.

Multi-simplicial objects are a good environment for the definition of higher categorical structures because there are natural candidates for the compositions given by the Segal maps.

We will introduce three Segal-type models, denoted collectively Seg_n

Let $X \in [\Delta^{op}, \mathcal{C}]$ be a simplicial object in a category $\mathcal{C}$ with pullbacks. Denote $X[k] = X_k$.

For each $k \geq 2$, let $\nu_i : X_k \to X_1$, $\nu_j = X(r_j)$, $r_j(0) = j - 1$, $r_j(1) = j$



There is a unique map, called Segal map

$$\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \ .$$

Segal maps and internal categories

- There is a nerve functor

$$N : \mathrm{Cat}\, \mathcal{C} \to [\Delta^{op}, \mathcal{C}]$$

$X \in \mathrm{Cat}\, \mathcal{C}$

$$NX \qquad \cdots X_1 \times_{X_0} X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow}\; X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}\; X_0$$

Fact: $X \in [\Delta^{op}, \mathcal{C}]$ is the nerve of an internal category in $\mathcal{C}$ if and only if all the Segal maps $\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$. are isomorphisms.

Multi-simplicial objects

- Let $\Delta^{n^{op}} = \Delta^{op} \times \overset{n}{\cdots} \times \Delta^{op}$.
- Multi-simplicial objects in $\mathcal{C}$ are functors $[\Delta^{n^{op}}, \mathcal{C}]$.
- They have n different simplicial directions and every n-fold simplicial object in $\mathcal{C}$ is a simplicial object in $(n-1)$-fold simplicial objects in $\mathcal{C}$ in n possible ways:

$$[\Delta^{n^{op}}, \mathcal{C}] \underset{\xi_k}{\cong} [\Delta^{op}, [\Delta^{n-1^{op}}, \mathcal{C}]] \qquad 1 \leq k \leq n$$

Thus for each $X \in [\Delta^{n^{op}}, \mathcal{C}]$ we have Segal maps in each of the n simplicial directions.

Definition

n-Fold categories are defined inductively by

$$\mathrm{Cat}^0 = \mathrm{Set}$$
$$\mathrm{Cat}^n = \mathrm{Cat}(\mathrm{Cat}^{n-1})$$

Definition

Strict n-categories are defined inductively by

$$0\text{-}\mathrm{Cat} = \mathrm{Set}$$
$$n\text{-}\mathrm{Cat} = ((n-1)\text{-}\mathrm{Cat})\text{-}\mathrm{Cat}$$

Multi-simplicial descriptions

- By iterating the nerve construction, we obtain fully faithful multinerve functors

$$N_{(n)} : \mathrm{Cat}^n \to [\Delta^{n^{op}}, \mathrm{Set}], \qquad N_{(n)} : n\text{-}\mathrm{Cat} \to [\Delta^{n^{op}}, \mathrm{Set}]$$
$$J_n : \mathrm{Cat}^n \to [\Delta^{{n-1}^{op}}, \mathrm{Cat}], \qquad J_n : n\text{-}\mathrm{Cat} \to [\Delta^{{n-1}^{op}}, \mathrm{Cat}] \,.$$

- We next characterize the essential image of these multinerve functors. This amounts to describing strict *n*-categories and *n*-fold categories multi-simplicially.

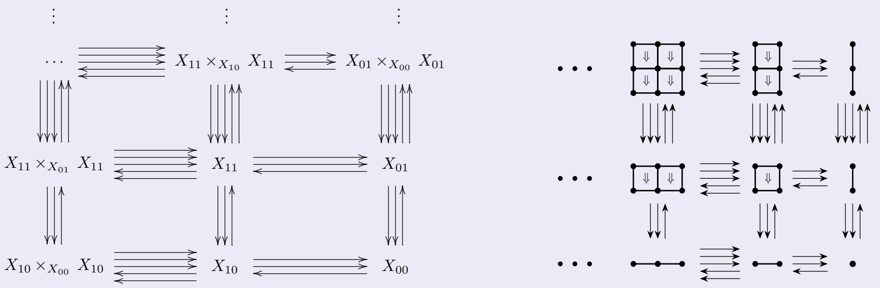
These descriptions facilitate the geometric intuition of how to modify the structure to build weak models.

An *n-fold category* is $X \in [\Delta^{{n-1}^{op}}, \mathrm{Cat}] \hookrightarrow [\Delta^{n^{op}}, \mathrm{Set}]$ such that the Segal maps in all directions are isomorphisms.

Note that $\mathrm{Cat}^{\mathrm{n}} \hookrightarrow [\Delta^{op}, \mathrm{Cat}^{\mathrm{n}-1}]$.

Let's illustrate the cases $n = 2, 3$.

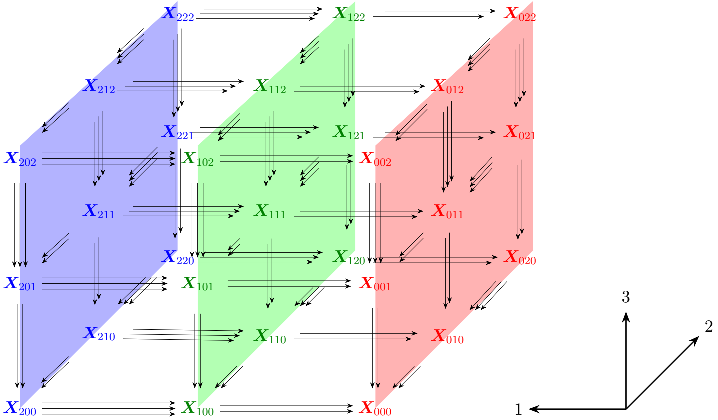
Example: double categories



Corner of the 3-fold nerve of a 3-fold category X

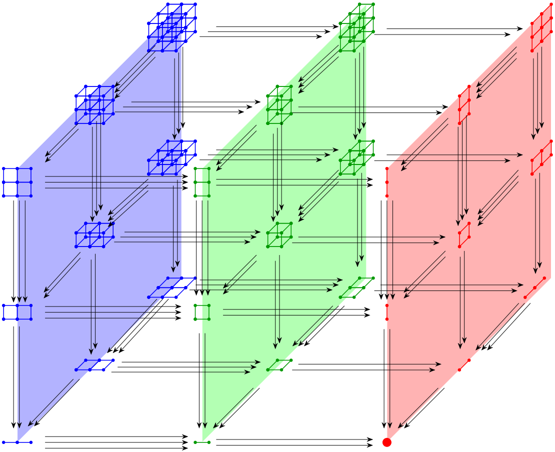
In the following picture, $X \in \mathsf{Cat}^3$ thus for all $i, j, k \in \Delta^{op}$

$$X_{2jk} \cong X_{1jk} \times_{X_{0jk}} X_{1jk}, \; X_{i2k} \cong X_{i1k} \times_{X_{i0k}} X_{i1k}, \; X_{ij2} \cong X_{ij1} \times_{X_{ij0}} X_{ij1} \; .$$



Geometric picture of the 3-fold nerve of a 3-fold category X

$$X \in \text{Cat}^3 \xhookrightarrow{N_{(3)}} [\Delta^{3^{op}}, \text{Set}]$$



Strict n-categories multi-simplicially

A strict n-category is $X \in [\Delta^{n-1^{op}}, \mathrm{Cat}] \hookrightarrow [\Delta^{n^{op}}, \mathrm{Set}]$ such that

i) *Segal condition*: The Segal maps in all directions are isomorphisms.

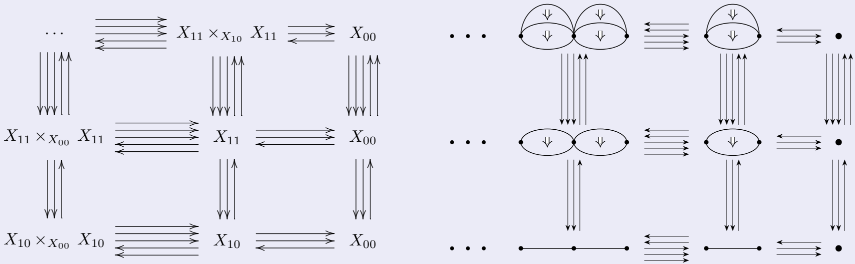
ii) *Globularity condition*: $X_0 \in [\Delta^{n-2^{op}}, \mathrm{Cat}]$ and $X_{k_1 \ldots k_r 0} \in [\Delta^{n-r-2^{op}}, \mathrm{Cat}]$ are constant functors taking value in a discrete category for all $1 \leq r \leq n-2$ and all $(k_1, \ldots, k_r) \in \Delta^{r^{op}}$.

Strict n-categories multi-simplicially, cont.

- The sets underlying the discrete structures X_0, (resp. $X_{\underset{r}{1\dots1}0}$) correspond to the sets of 0-cells (resp. r-cells) for $1 \leq r \leq n-2$.
- The set of $(n-1)$ (resp. n)-cells is given by $ob(X_{\underset{n-1}{1\dots1}})$ (resp.$mor(X_{\underset{n-1}{1\dots1}})$).
- Note that n-Cat $\hookrightarrow [\Delta^{op}, (n-1)\text{-Cat}]$.

Let's illustrate the cases $n = 2, 3$.

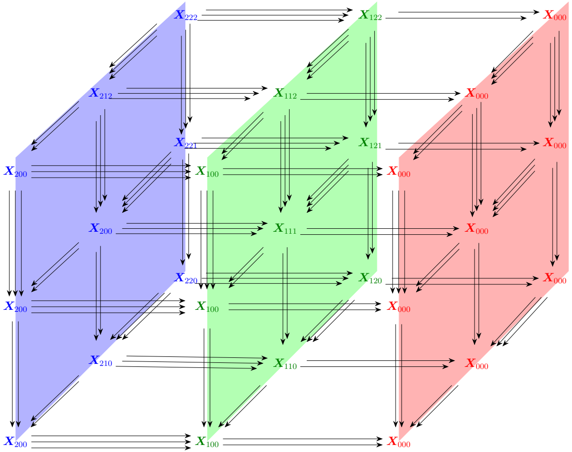
Example: strict 2-categories



Corner of the 3-fold nerve of a strict 3-category X

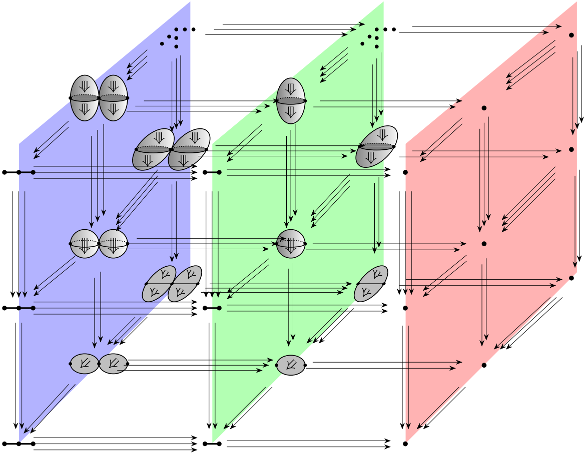
$$X_{2jk} \cong X_{1jk} \times_{X_{0jk}} X_{1jk}, \;\; X_{i2k} \cong X_{i1k} \times_{X_{i0k}} X_{i1k}, \;\; X_{ij2} \cong X_{ij1} \times_{X_{ij0}} X_{ij1} \, .$$

$$X \in \; 3-\mathrm{Cat} \xhookrightarrow{N_{(3)}} [\Delta^{3^{op}}, \mathrm{Set}]$$



Geometric picture of the 3-fold nerve of a strict 3-category X

$$3\text{-Cat} \xhookrightarrow{N_{(3)}} [\Delta^{3^{op}}, \text{Set}]$$



Hom$(n-1)$-category and truncation functor

- Hom $(n-1)$-category. For each $a, b \in X_0$, $X(a,b) \in (n-1)$-Cat is the fiber at (a,b) of $X_1 \xrightarrow{(\partial_0,\partial_1)} X_0 \times X_0$.

- Truncation functor $p^{(n-1)} : n\text{-Cat} \hookrightarrow [\Delta^{n-1^{op}}, \text{Cat}] \to (n-1)\text{-Cat}$

$$(p^{(n-1)}X)_{k_1 \dots k_{n-1}} = pX_{k_1 \dots k_{n-1}}$$

  where $p : \text{Cat} \to \text{Set}$ is the isomorphism classes of object functor.

The truncation functor divides out by the highest dimensional invertible cells.

n-Equivalences

- A 1-equivalence is an equivalence of categories.

  Suppose, inductively, that we defined $(n-1)$-equivalences.
  A morphism $F : X \to Y$ in n-Cat is an *n-equivalence* if

  (a) For all $a, b \in X_0$, $F(a,b) : X(a,b) \to Y(Fa, Fb)$ is a $(n-1)$-equivalence.

  (b) $p^{(n-1)}F$ is a $(n-1)$-equivalence.

This definition is a higher dimensional generalization of a functor which is fully faithful and essentially surjective on objects.

Weakening the multi-simplicial definition of strict n-categories

n-Cat	$\mathrm{Seg_n}$
Multi-simplicial embedding	$\mathrm{Seg_n} \hookrightarrow [\Delta^{n-1^{op}}, \mathrm{Cat}] \hookrightarrow [\Delta^{n^{op}}, \mathrm{Set}]$
Inductive definition	$\mathrm{Seg_1} = \mathrm{Cat}$, $\mathrm{Seg_n} \hookrightarrow [\Delta^{op}, \mathrm{Seg_{n-1}}]$
Truncation functor	$p^{(n-1)} : \mathrm{Seg_n} \to \mathrm{Seg_{n-1}}$
Hom $(n-1)$-category	Similar definition
n-equivalences	Same definition
Globularity condition	Different (weak globularity)
Segal condition	Different (induced Segal condition)

The idea of homotopically discrete n-fold category

- A homotopically discrete category is an equivalence relation.
- Given $X \in \mathsf{Cat}_{\mathsf{hd}}$, there is a functor $X \to pX$.

A homotopically discrete n-fold category is an n-fold category suitably equivalent to a discrete one both 'globally' and in each simplicial dimension.

Definition

Let $\mathsf{Cat}^0_{\mathsf{hd}} = \mathsf{Set}$.

Suppose, inductively, we defined the subcategory $\mathsf{Cat}^{n-1}_{\mathsf{hd}} \subset \mathsf{Cat}^{n-1}$ of homotopically discrete $(n-1)$-fold categories. We say that the n-fold category $X \in \mathsf{Cat}^n \hookrightarrow [\Delta^{{n-1}^{op}}, \mathsf{Cat}]$ is homotopically discrete if:

- X is a levelwise equivalence relation.
- $p^{(n-1)} X \in \mathsf{Cat}^{n-1}_{\mathsf{hd}}$.

We denote $\mathsf{Cat}^1_{\mathsf{hd}} = \mathsf{Cat}_{\mathsf{hd}}$.

The discretization map

Definition

- Given $X \in \mathrm{Cat}^{\mathrm{n}}_{\mathrm{hd}}$ let $\gamma_X^{(n-1)} : X \to p^{(n-1)}X$ be the morphism given levelwise for each $\underline{s} \in \Delta^{n-1^{op}}$ by

$$(\gamma_X^{(n-1)})_{\underline{s}} : X_{\underline{s}} \to pX_{\underline{s}}$$

- The discretization map is the composite

$$\gamma_{(n)} : X \xrightarrow{\gamma^{(n-1)}} p^{(n-1)}X \xrightarrow{\gamma^{(n-2)}} p^{(n-2)}p^{(n-1)}X \to \cdots \xrightarrow{\gamma^{(0)}} X^d$$

where $X^d = p^{(0)}p^{(1)}...p^{(n-1)}X$ is called discretization of X.

Let $X \in \mathsf{Seg_n}$.

- Weak globularity condition: $X_0,\ X_{k_1,\ldots,k_r 0}$ are homotopically discrete for all $1 \le r \le n-2$ and all $(k_1,\ldots,k_r) \in \Delta^{r^{op}}$.

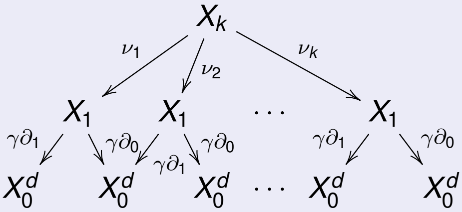
The sets underlying the discrete structures X_0^d, $X^d_{\underset{r}{1\ldots1}0}$ correspond to the sets of r-cells for $0 \le r \le n-2$. The sets of $(n-1)$ (resp. n)-cells correspond to $ob(X_{\underset{n-1}{1\ldots1}})$ (resp.$mor(X_{\underset{n-1}{1\ldots1}})$).

- For each $a, b \in X_0^d$, let $X(a,b)$ be the fiber at (a,b) of

$$X_1 \xrightarrow{(\partial_0,\partial_1)} X_0 \times X_0 \xrightarrow{\gamma\times\gamma} X_0^d \times X_0^d\ .$$

Induced Segal maps condition

Given $X \in \mathrm{Seg_n} \subset [\Delta^{op}, \mathrm{Seg_{n-1}}]$, consider the commuting diagram



where $k \geq 2$, $\nu_j = X(r_j)$, $r_j(0) = j-1$, $r_j(1) = j$. The corresponding induced Segal map

$$\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \ .$$

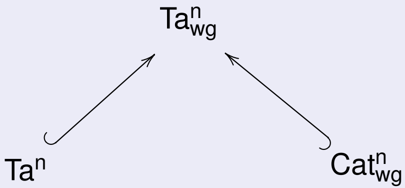
is required to be an $(n-1)$-equivalences in $\mathrm{Seg_{n-1}}$ for each $k \geq 2$.

Summary of main common features of $\mathsf{Seg_n}$.

- Multi-simplicial embeddings $\mathsf{Seg_n} \hookrightarrow [\Delta^{{n-1}^{op}}, \mathsf{Cat}] \hookrightarrow [\Delta^{{n}^{op}}, \mathsf{Set}]$.
- Inductive definition $\mathsf{Seg_1} = \mathsf{Cat}$, $\mathsf{Seg_n} \hookrightarrow [\Delta^{op}, \mathsf{Seg_{n-1}}]$.
- Weak globularity condition.
- Functor $p^{(n-1)} : \mathsf{Seg_n} \to \mathsf{Seg_{n-1}}$
- n-Equivalences.
- $(n-1)$-Equivalences of the induced Segal maps for each $k \geq 2$

$$\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \ .$$

We discuss three Segal-type models,collectively denoted $\mathrm{Seg_n}$



$\mathrm{Ta^n}$ Tamsamani *n*-categories [Tamsamani and Simpson]

$\mathrm{Cat^n_{wg}} \subset \mathrm{Cat^n}$ weakly globular *n*-fold categories [P.]

$\mathrm{Ta^n_{wg}}$ weakly globular Tamsamani *n*-categories [P.]

Respective functor $p^{(n-1)} : \mathrm{Seg_n} \to \mathrm{Seg_{n-1}}$ for each model.

Three different models corresponding to different behavior of:

Induced Segal maps $\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$

Segal maps $\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$

	$X_0, X_{k_1,\dots,k_r 0}$	$\hat{\mu}_k$	η_k
Ta^n	discrete	$(n-1)$-eq	$(n-1)$-eq
Cat^n_{wg}	homotopically discrete	$(n-1)$-eq	isomorphisms
Ta^n_{wg}	homotopically discrete	$(n-1)$-eq	-

Main results [P. and Pronk $n = 2$, P. for $n > 2$]

Theorem A. There is a functor *rigidification*

$$Q_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$$

and for each $X \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ an n-equivalence natural in X

$$s_n(X) : Q_n X \to X.$$

Theorem B. There is a functor *discretization*

$$Disc_n : \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Ta}^{\mathrm{n}}$$

and, for each $X \in \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$, a zig-zag of n-equivalences in $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ between X and $Disc_n X$.

Main results, cont.

Theorem C. The functors

$$Q_n : \mathrm{Ta^n} \leftrightarrows \mathrm{Cat^n_{wg}} : Disc_n$$

induce an equivalence of categories after localization with respect to the n-equivalences

$$\mathrm{Ta^n}/\sim^n \simeq \mathrm{Cat^n_{wg}}/\sim^n \; .$$

Theorem D. There is an equivalence of categories

$$\mathrm{GCat^n_{wg}}/\sim^n \simeq \mathrm{Ho}\,(n\text{-types}) \; .$$

where $\mathrm{GCat^n_{wg}} \subset \mathrm{Cat^n_{wg}}$ is the subcategory of groupoidal weakly globular n-fold categories.

Using pseudo-functors to rigidify $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$

We identify a subcategory

$$\mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \subset \mathrm{Ps}[\Delta^{n-1^{op}}, \mathrm{Cat}]$$

of Segalic pseudo-functors with the property that the strictification functor $St\ : \mathrm{Ps}[\Delta^{n-1^{op}}, \mathrm{Cat}] \rightarrow [\Delta^{n-1^{op}}, \mathrm{Cat}]$ restricts to

$$\mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \subset [\Delta^{n-1^{op}}, \mathrm{Cat}].$$

We will then build the rigidification functor Q_n as a composite

$$Q_n : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \rightarrow \mathrm{SegPs}[\Delta^{n-1^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}.$$

Segal maps for pseudo-functors.

- Notation:

  $\underline{k} = (k_1, \ldots, k_{n-1}) \in \Delta^{n-1^{op}}$, $1 \leq i \leq n-1$

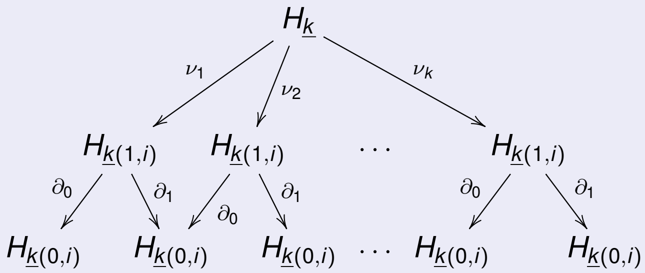
  $\underline{k}(1,i) = (k_1, \ldots, k_{i-1}, 1, k_{i+1}, \ldots, k_{n-1})$

  $\underline{k}(0,i) = (k_1, \ldots, k_{i-1}, 0, k_{i+1}, \ldots, k_{n-1})$

- Let $H \in \mathrm{Ps}[\Delta^{n-1^{op}}, \mathrm{Cat}]$ be such that $H_{\underline{k}(0,i)}$ is discrete for all $\underline{k} \in \Delta^{n-1^{op}}$ and all $1 \leq i \leq n-1$.

For pseudo-functors H satisfying this condition we can define Segal maps as follows.

The following diagram in Cat commutes



Hence there is a unique Segal map for all $k_i \geq 0$

$$H_{\underline{k}} \to H_{\underline{k}(1,i)} \times_{H_{\underline{k}(0,i)}} \overset{k_i}{\cdots} \times_{H_{\underline{k}(0,i)}} H_{\underline{k}(1,i)} \; .$$

The functor $p^{(n-1)}$

Definition

We denote by

$$p^{(n-1)} : \text{Ps}[\Delta^{{n-1}^{op}}, \text{Cat}] \to [\Delta^{{n-1}^{op}}, \text{Set}]$$

the functor $(p^{(n-1)}X)_{\underline{k}} = pX_{\underline{k}}$ for $X \in \text{Ps}[\Delta^{{n-1}^{op}}, \text{Cat}]$ and $\underline{k} \in \Delta^{{n-1}^{op}}$.

Definition

Define $H \in \mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \subset \mathrm{Ps}[\Delta^{{n-1}^{op}}, \mathrm{Cat}]$ if

i) $H_{\underline{k}(0,i)}$ is discrete for all $\underline{k} \in \Delta^{{n-1}^{op}}$ and $1 \leq i \leq n-1$.

ii) All Segal maps are isomorphisms.

iii) The functor $p^{(n-1)} : \mathrm{Ps}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \to [\Delta^{{n-1}^{op}}, \mathrm{Set}]$ restricts to a functor

$$p^{(n-1)} : \mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \to \mathrm{Cat}^{\mathrm{n-1}}_{\mathrm{wg}} .$$

The idea is to add an extra pseudo-simplicial dimension to $\mathrm{Cat}^{\mathrm{n-1}}_{\mathrm{wg}}$ in such a way that Segal maps can be defined and are isomorphims.

From Segalic pseudo-functors to weakly globular n-fold categories

Theorem

The strictification functor

$$St : \mathrm{Ps}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \to [\Delta^{{n-1}^{op}}, \mathrm{Cat}]$$

restricts to a functor

$$St : \mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$$

Next we build a functor $\mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}]$, distinguishing the cases $n = 2$ and $n > 2$.

From $\mathsf{Ta}^2_{\mathsf{wg}}$ to pseudo-functors

- By definition, $X \in \mathsf{Ta}^2_{\mathsf{wg}}$ if $X \in [\Delta^{op}, \mathsf{Cat}]$ is such that

$$X_0 \in \mathsf{Cat}_{\mathsf{hd}}, \qquad X_k \simeq X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \quad k \geq 2.$$

- Let

$$(Tr_2 X)_k = \begin{cases} X_0^d, & k = 0 \\ X_1, & k = 1 \\ X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1, & k > 1 \end{cases}$$

  Then $X_k \simeq (Tr_2 X)_k$ for all k.

By transport of structure $Tr_2 X \in [ob(\Delta^{op}), \mathsf{Cat}]$ lifts to a pseudo-functor $Tr_2 X \in \mathsf{Ps}[\Delta^{op}, \mathsf{Cat}]$, which is Segalic.

Definition

Let Q_2 be the composite

$$Q_2 : \mathrm{Ta}^2_{\mathrm{wg}} \xrightarrow{Tr_2} \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^2_{\mathrm{wg}}$$

and let $s_2(X) : Q_2 X = St\, Tr_2 X \to X$ correspond by adjointness to $t_2(X) : Tr_2 X \to X$.

- One can show that $s_2(X)$ is a 2-equivalence.

From Ta^{n}_{wg} to pseudo-functors

- The case $n > 2$ is more complex, since the induced Segal maps of $X \in \mathsf{Ta}^{n}_{wg}$ are $(n-1)$-equivalences but not, in general, levelwise equivalences of categories.

We identify a subcategory $\mathsf{LTa}^{n}_{wg} \subset \mathsf{Ta}^{n}_{wg}$ and functors

$$\mathsf{Ta}^{n}_{wg} \xrightarrow{P_n} \mathsf{LTa}^{n}_{wg} \xrightarrow{Tr_n} \mathsf{SegPs}[\Delta^{{n-1}^{op}}, \mathsf{Cat}]\ .$$

- The functor Tr_n is built using transport of structure in a way formally analogous to the case $n = 2$.

The functor $q^{(n-1)}$

Let $q : \mathrm{Cat} \to \mathrm{Set}$ be the connected components functor.

Proposition

- *The functor* $q^{(n-1)} : [\Delta^{n^{op}}, \mathrm{Set}] \to [\Delta^{n-1^{op}}, \mathrm{Set}]$ *obtained by applying q levelwise restricts to a functor* $q^{(n-1)} : \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Ta}^{\mathrm{n-1}}_{\mathrm{wg}}$.
- *For each* $X \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$, *there is a map natural in* X

$$\gamma^{(n-1)} : X \to q^{(n-1)} X \; .$$

The functor $q^{(n-1)}$ divides out by the highest dimensional cells. Think of $q^{(n-1)}$ as a 'categorical Postnikov functor'.

Strategy in building the rigidification functor Q_n

- We show that, if $X \in \mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}}$ is such that $q^{(n-1)}X$ can be approximated up to $(n-1)$-equivalence with an object of $\mathsf{Cat}^{\mathsf{n-1}}_{\mathsf{wg}}$, then X can be approximated up to an n-equivalence with an object of $\mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$.

- This property is used to construct the functor $P_n : \mathsf{Ta}^{\mathsf{n}}_{\mathsf{wg}} \to \mathsf{LTa}^{\mathsf{n}}_{\mathsf{wg}}$.

The functor P_n

- Suppose, inductively, that we defined $Q_{n-1} : \mathrm{Ta}^{\mathrm{n-1}}_{\mathrm{wg}} \to \mathrm{Cat}^{\mathrm{n-1}}_{\mathrm{wg}}$ and and the $(n-1)$-equivalence $s_{n-1}Y : Q_{n-1}Y \to Y$ for each $Y \in \mathrm{Ta}^{\mathrm{n-1}}_{\mathrm{wg}}$.

Given $X \in \mathrm{Ta}^{\mathrm{n}}_{\mathrm{wg}}$ let $P_n X$ be the pullback in $[\Delta^{n-1^{op}}, \mathrm{Cat}]$

$$\begin{array}{ccc} P_n X & \xrightarrow{\quad w_n(X) \quad} & X \\ \big\downarrow & & \big\downarrow{\scriptstyle \gamma^{(n-2)}} \\ Q_{n-1}q^{(n-1)}X & \xrightarrow[\quad s_{n-1}(q^{(n-1)}X) \quad]{} & q^{(n-1)}X \end{array}$$

Then $P_n X \in \mathrm{LTa}^{\mathrm{n}}_{\mathrm{wg}}$ and $w_n(X)$ is an n-equivalence.

Definition

Let Q_2 be the composite

$$Q_2 : \mathrm{Ta}^2_{\mathrm{wg}} \xrightarrow{Tr_2} \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^2_{\mathrm{wg}}$$

Define Q_n for $n > 2$ to be the composite

$$Q_n : \mathrm{Ta}^n_{\mathrm{wg}} \xrightarrow{P_n} \mathrm{LTa}^n_{\mathrm{wg}} \xrightarrow{Tr_n} \mathrm{SegPs}[\Delta^{{n-1}^{op}}, \mathrm{Cat}] \xrightarrow{St} \mathrm{Cat}^n_{\mathrm{wg}}.$$

Summary of rigidification process

Definition:
$\mathsf{Cat}^n_{\mathsf{hd}}$

Definitions:
$\mathsf{Ta}^n_{\mathsf{wg}}$, Ta^n, $\mathsf{Cat}^n_{\mathsf{wg}}$

Definition:
$\mathsf{LTa}^n_{\mathsf{wg}}$

Theorem:
$P_n : \mathsf{Ta}^n_{\mathsf{wg}} \to \mathsf{LTa}^n_{\mathsf{wg}}$

Definition:
$\mathsf{SegPs}[\Delta^{{n-1}^{op}}, \mathsf{Cat}]$

Theorem:
$St : \mathsf{SegPs}[\Delta^{{n-1}^{op}}, \mathsf{Cat}] \to \mathsf{Cat}^n_{\mathsf{wg}}$

Theorem: Rigidification functor

$Q_2 : \mathsf{Ta}^2_{\mathsf{wg}} \xrightarrow{Tr_2} \mathsf{SegPs}[\Delta^{op}, \mathsf{Cat}] \xrightarrow{St} \mathsf{Cat}^2_{\mathsf{wg}}$

For $n > 2$

$Q_n : \mathsf{Ta}^n_{\mathsf{wg}} \xrightarrow{P_n} \mathsf{LTa}^n_{\mathsf{wg}} \xrightarrow{Tr_n}$
$\mathsf{SegPs}[\Delta^{{n-1}^{op}}, \mathsf{Cat}] \xrightarrow{St} \mathsf{Cat}^n_{\mathsf{wg}}$

The idea of the discretization functor

- The idea of $Disc_n : \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} \to \mathrm{Ta}^{\mathrm{n}}$ is to replace the homotopically discrete sub-structures in $\mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}$ by their discretizations.

- This recovers the globularity condition, but at the expenses of the Segal maps, which from being isomorphisms become $(n-1)$-equivalences.

The case $n = 2$

- Let $X \in \mathrm{Cat}^2_{\mathrm{wg}}$, then $X_0 \in \mathrm{Cat}_{\mathrm{hd}}$.
- Choose a section $\gamma' : X_0^d \to X_0$ of $\gamma : X_0 \to X_0^d$.

Let $D_0 X \in [\Delta^{op}, \mathrm{Cat}]$ be given by

$$\cdots \quad X_1 \times_{X_0} X_1 \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; X_1 \; \substack{\xrightarrow{\gamma\partial_0} \\ \xrightarrow{\gamma\partial_1} \\ \xleftarrow[\sigma\gamma']{}} \; X_0^d$$

The Segal maps of $D_0 X$ for each $k \geq 2$

$$X_k = X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$$

are equivalences of categories and X_0^d is discrete. Thus $D_0 X \in \mathrm{Ta}^2$.

The case $n = 2$, cont.

- Given a map $f : X \to Y$ in $\mathrm{Cat}^2_{\mathrm{wg}}$, we have a pseudo-commuting diagram

$$\begin{array}{ccc} X_0^d & \longrightarrow & Y_0^d \\ {\scriptstyle \gamma'_{X_0}}\downarrow & & \downarrow{\scriptstyle \gamma'_{Y_0}} \\ X_0 & \longrightarrow & Y_0 \end{array}$$

  for given choices of sections $\gamma'_{X_0}, \gamma'_{Y_0}$.

Therefore the corresponding map $D_0 X \to D_0 Y$ is in $\mathrm{Ps}[\Delta^{op}, \mathrm{Cat}]$. That is

$$D_0 : \mathrm{Cat}^2_{\mathrm{wg}} \to (\mathrm{Ta}^2)_{\mathrm{ps}} \; .$$

Overall strategy

- To remedy this problem we introduce the category FCat^{n}_{wg} which exhibits functorial sections to the discretization maps of the homotopically discrete substructures in Cat^{n}_{wg}.
- Because of this property of FCat^{n}_{wg}, the discretization process can be done functorially, using an iteration of the above idea, via a functor $D_n : \mathrm{FCat}^{n}_{wg} \to \mathrm{Ta}^{n}$.
- We show that we can approximate any object of Cat^{n}_{wg} with an n-equivalent object of FCat^{n}_{wg} via a functor $G_n : \mathrm{Cat}^{n}_{wg} \to \mathrm{FCat}^{n}_{wg}$.
- $Disc_n$ is defined as the composite $\mathrm{Cat}^{n}_{wg} \xrightarrow{G_n} \mathrm{FCat}^{n}_{wg} \xrightarrow{D_n} \mathrm{Ta}^{n}$.

The main comparison result

Theorem

The functors

$$Q_n : \mathrm{Ta}^{\mathrm{n}} \leftrightarrows \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}} : Disc_n$$

induce an equivalence of categories after localization with respect to the n-equivalences

$$\mathrm{Ta}^{\mathrm{n}}/\sim^n \simeq \mathrm{Cat}^{\mathrm{n}}_{\mathrm{wg}}/\sim^n$$

Groupoidal weakly globular n-fold categories

Definition

Define $\mathrm{GCat}^{n}_{\mathrm{wg}} \subset \mathrm{Cat}^{n}_{\mathrm{wg}}$ inductively

$n = 1 \quad \mathrm{GCat}^{1}_{\mathrm{wg}} = \mathrm{Gpd}$

Suppose we defined $\mathrm{GCat}^{n-1}_{\mathrm{wg}}$.

$X \in \mathrm{GCat}^{n}_{\mathrm{wg}} \subset \mathrm{Cat}^{n}_{\mathrm{wg}}$ if

i) for all $a, b \in X^{d}_{0}$, $X(a,b) \in \mathrm{GCat}^{n-1}_{\mathrm{wg}}$.

ii) $p^{(n-1)}X \in \mathrm{GCat}^{n-1}_{\mathrm{wg}} \subset \mathrm{Cat}^{n-1}_{\mathrm{wg}}$.

From the comparison theorem between $\mathsf{Cat}^{\mathsf{n}}_{\mathsf{wg}}$ and Ta^{n} we obtain

Theorem

There is an equivalence of categories

$$\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}/\sim^{n}\ \simeq\ Ho(n\text{-}types)\ .$$

Note: An explicit description of the fundamental *n*-groupoid functor

$$\mathsf{Top} \to \mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}$$

is given by [Blanc and P.,2015].

Overall Summary

The three Segal-type models and Segalic pseudo-functors

Definition: Cat^n_{hd}

Definition: $\mathsf{Ta}^n_{wg}, \mathsf{Ta}^n$

Definition: Cat^n_{wg}

Definition: $\mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$

Theorem

$St : \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}] \to \mathsf{Cat}^n_{wg}$

Rigidification of weakly globular Tamsamani *n*-categories

Definition: LTa^n_{wg}

Theorem

$Tr_n : \mathsf{LTa}^n_{wg} \to \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}]$

Theorem: Rigidification functor

$Q_n : \mathsf{Ta}^n_{wg} \xrightarrow{P_n} \mathsf{LTa}^n_{wg} \xrightarrow{Tr_n}$

$\to \mathsf{SegPs}[\Delta^{n-1^{op}}, \mathsf{Cat}] \xrightarrow{St} \mathsf{Cat}^n_{wg}$

Weakly globular *n*-fold categories as a model of weak *n*-categories

Definition: FCat^n_{wg}

Definitions: GCat^n_{wg}, GTa^n_{wg}, GTa^n

Theorem : Discretization functor

$Disc_n : \mathsf{Cat}^n_{wg} \to \mathsf{Ta}^n$

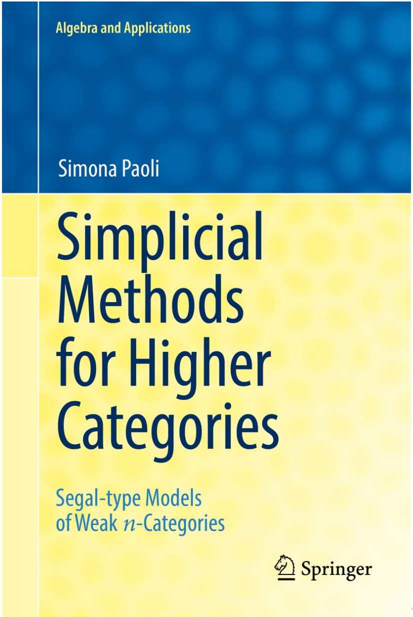
Theorem: $\mathsf{Ta}^n/\sim^n \;\simeq\; \mathsf{Cat}^n_{wg}/\sim^n$

Theorem: $\mathsf{GCat}^n_{wg}/\sim^n \;\simeq\; \mathsf{Ho}(n\text{-types})$

Further directions

- Postnikov systems of simplicial categories.
- Model category theoretic approaches.
- Weak globularity in the (∞, n) context.
- Weak units.
- Comparison of Segalic and operadic approaches.

Book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