A formal category theory for ∞-categories

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defn (Lawvere–Tierney) An elementary 1-topos is

(0) a cartesian closed 1-category that has

(1) finite limits and

(2) a subobject classifier.

defn (Weber, after Street) A 2-topos is

(0) a cartesian closed 2-category that has

(1) finite limits,

(γ) a duality involution, and

(2) a classifying left fibration.

Fix two Grothendieck universes $\mathcal{U} < \mathcal{U}'$.

Ex The 1-category **Set** of $\mathcal{U}$-small sets is a **1-topos**.

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "\top"] \\ A \arrow[r, "\chi_S"'] & \Omega \end{tikzcd}$$

$\Omega = \{\top, \bot\}$ classifies subobjects

Ex The 2-category **CAT** of $\mathcal{U}'$-small categories is a **2-topos**.

$$\begin{tikzcd} \int F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{S}et_* \arrow[d, "U"] \\ A \arrow[r, "F"'] & \mathcal{S}et \end{tikzcd}$$

classifies **left fibrations** with $\mathcal{U}$-small fibers

(discrete opfibrations)

Main result (R-Verity) A weakened version of these 2-topos axioms are satisfied by $\mathcal{CAT}_\infty$, the 2-category of ∞-categories.

Here and elsewhere ∞-category is shorthand for $(\infty,1)$-category, a category weakly enriched in ∞-groupoids/homotopy types.

PLAN (0) a cartesian closed 2-category $\mathcal{CAT}_\infty$

(1) finite limits in $\mathcal{CAT}_\infty$

(γ) a duality involution on $\mathcal{CAT}_\infty$

(2) a classifying left fibration for $\mathcal{CAT}_\infty$

Part 0: a cartesian closed 2-category $\mathcal{CAT}_\infty$

We can use various models of ∞-categories to define the 2-category $\mathcal{CAT}_\infty$ and the results will be biequivalent.

Theorem (Joyal, Rezk, Joyal-Tierney, Verity) The 1-categories of quasi-categories, complete Segal spaces, Segal categories, and 1-complicial sets are cartesian closed.

$\rightsquigarrow$ For $\mathcal{K} = q\mathcal{C}at$, $\mathcal{CSS}$, $\mathcal{S}egal$, or $1\text{-}\mathcal{C}omp$, $\mathcal{K}$ is a $\mathcal{K}$-category:

$$C^{A\times B} \cong (C^B)^A \in \mathcal{K}$$

The homotopy category functor $\mathcal{K} \xrightarrow{h_0} \mathcal{CAT}$ preserves products $\rightsquigarrow$ $\mathcal{K}$ is a cartesian closed 2-category with $\mathrm{fun}(A,B) := h_0(B^A)$.

$\rightsquigarrow$ this defines the 2-category $\mathcal{CAT}_\infty$

Part 1: weak finite limits in $\mathcal{CAT}_\infty$

Prop. Given $C \xrightarrow{g} A \xleftarrow{f} B$ in $\mathcal{CAT}_\infty$ there exists a weak comma object

$$
\begin{tikzcd}
 & \mathrm{Hom}_A(f,g) \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] \arrow[dd, phantom, "\overset{\phi}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
$$

constructed by

$$
\begin{tikzcd}
\mathrm{Hom}_A(f,g) \arrow[r, "\phi"] \arrow[d, "(\mathrm{cod},\mathrm{dom})"'] \arrow[dr, phantom, "\lrcorner" very near start] & A^2 \arrow[d, two heads] \\
C \times B \arrow[r, "g \times f"'] & A \times A
\end{tikzcd}
\quad \text{in } \mathcal{K}
$$

such that

$$
\left\{
\begin{tikzcd}
 & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, phantom, "\overset{\alpha}{\Leftarrow}"] & \\
C \arrow[dr, "g"'] & & B \arrow[dl, "f"] \\
 & A &
\end{tikzcd}
\right\}
\longleftrightarrow
\left\{
\begin{tikzcd}
 & X \arrow[dl, "c"'] \arrow[dr, "b"] \arrow[dd, dashed, "\ulcorner \alpha \urcorner"] & \\
C & & B \\
 & \mathrm{Hom}_A(f,g) \arrow[ul, "\mathrm{cod}"] \arrow[ur, "\mathrm{dom}"'] &
\end{tikzcd}
\right\}
$$

up to iso over $C \times B$.

Cor. $\mathrm{Hom}_A(f,g)$ is unique up to *fibered equivalence*.

defn. $A \xrightarrow{k} B$ is **fully faithful** iff $A^{\mathbb{2}} \xrightarrow[\sim]{\ulcorner \mathrm{id}_k \urcorner} \mathrm{Hom}_A(k,k)$.

defn. A **pointwise right Kan extension** between ∞-categories is

$$\begin{tikzcd} A \arrow[rr, "k"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Leftarrow \nu"] & B \arrow[dl, "r"] \\ & C & \end{tikzcd}$$

such that

$$\begin{tikzcd} \mathrm{Hom}_B(b,k) \arrow[rr, two heads] \arrow[d] & \arrow[d, phantom, "\Swarrow \phi"] & X \arrow[d, "b"] \\ A \arrow[rr, "k"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Leftarrow \nu"] & B \arrow[dl, "r"] \\ & C & \end{tikzcd}$$

is also a right Kan extension.

Prop. If k is fully faithful, then ν is invertible.

Prop. A right adjoint is fully faithful iff the counit is invertible.

Part γ: a duality involution on $\mathcal{CAT}_\infty$

Like $\mathcal{CAT}$, $\mathcal{CAT}_\infty$ has a 2-functor $\mathcal{CAT}_\infty^{co} \xrightarrow{(-)^\circ} \mathcal{CAT}_\infty$ that sends an ∞-category A to its opposite ∞-category A°, such that $(A^\circ)^\circ = A$.

But Weber's notion of duality involution requires more...

To state the full axiom we must introduce:

two-sided discrete fibrations aka modules between ∞-categories.

defn. A functor $E \overset{p}{\twoheadrightarrow} B$ between ∞-categories is a

left fibration iff $E \xrightarrow{2^{\ulcorner} id_p^{\urcorner}} \mathrm{Hom}_B(p, B)$ is an equivalence

right fibration iff $E \xrightarrow{2^{\ulcorner} id_p^{\urcorner}} \mathrm{Hom}_B(B, p)$

cocartesian fibration iff $E \xrightarrow{2^{\ulcorner} id_p^{\urcorner}} \mathrm{Hom}_B(p, B)$ admits a $\begin{cases} \text{lari} \\ \text{rari} \end{cases}$

cartesian fibration iff $E \xrightarrow{2^{\ulcorner} id_p^{\urcorner}} \mathrm{Hom}_B(B, p)$

Ex $A^2 \overset{dom}{\twoheadrightarrow} A$ / $A^2 \overset{cod}{\twoheadrightarrow} A$ is a $\begin{cases} \text{cartesian} \\ \text{cocartesian} \end{cases}$ fibration

Ex For $1 \xrightarrow{a} A$, $\mathrm{Hom}_A(A, a) \overset{dom}{\twoheadrightarrow} A$ / $\mathrm{Hom}_A(a, A) \overset{cod}{\twoheadrightarrow} A$ is a $\begin{cases} \text{right} \\ \text{left} \end{cases}$ fibration defn. A span $A \overset{q}{\twoheadleftarrow} E \overset{p}{\twoheadrightarrow} B$ of ∞-categories defines a **module from A to B** iff

- $A \overset{q}{\twoheadleftarrow} E$ is a cocartesian fibration over B
- $E \overset{p}{\twoheadrightarrow} B$ is a cartesian fibration over A
- the fibers of $E \xrightarrow{(q,p)} A \times B$ are ∞-groupoids

Ex any left fibration $A \overset{q}{\twoheadleftarrow} E \twoheadrightarrow 1$ or any right fibration $1 \twoheadleftarrow E \overset{p}{\twoheadrightarrow} B$

Ex $A \xleftarrow{\mathrm{cod}} A^{2} \xrightarrow{\mathrm{dom}} A$

$C \xleftarrow{\mathrm{cod}} \mathrm{Hom}_A(f,g) \xrightarrow{\mathrm{dom}} B$ for any $C \xrightarrow{g} A \xleftarrow{f} B$

$1 \longleftarrow \mathrm{Hom}_A(A,a) \xrightarrow{\mathrm{dom}} A$ for any $1 \xrightarrow{a} A$

$A \xleftarrow{\mathrm{cod}} \mathrm{Hom}_A(a,A) \longrightarrow 1$

Idea: a module $A \leftarrow\!\!\!\leftarrow E \twoheadrightarrow B$ encodes a homotopy coherent diagram $A \times B^{\circ} \to \infty\text{-Gpd}$ ... but so does a left fibration $F \twoheadrightarrow A \times B^{\circ}$ or a right fibration $G \twoheadrightarrow A^{\circ} \times B$

defn (Weber) A duality involution entails an involutive 2-functor $(-)^{\circ}$, contravariant on 2-cells, together with a pseudo-natural equivalence of categories

$$\left\{ \begin{array}{c} \text{modules from} \\ A \times B^{\circ} \text{ to } C \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{modules from} \\ A \text{ to } B \times C \end{array} \right\}$$

Our next task is to construct this for $\mathcal{CAT}_\infty$.

We work with the quasi-categorical model of ∞-categories.

defn. The quasi-category $A \wr A$ of twisted arrows in a quasi-category A has

$$\{\Delta[n] \to A \wr A\} \cong \{\Delta[n]^{\circ} \star \Delta[n] \to A\}$$

Consider also $A \wr^{\circ} A := (A \wr A)^{\circ}$

$$\{\Delta[n] \to A \wr^{\circ} A\} \cong \{\Delta[n] \star \Delta[n]^{\circ} \to A\}$$

Prop. $A \wr A \xrightarrow{(\mathrm{cod}, \mathrm{dom})} A \times A^{\circ}$ is a left fibration.

$A \wr^{\circ} A \xrightarrow{(\mathrm{cod}, \mathrm{dom})} A^{\circ} \times A$ is a right fibration.

Prop. For all $1 \xrightarrow{a} A$,

$$\begin{tikzcd} a \wr A \arrow[r, phantom, "\simeq"] \arrow[dr, "\mathrm{cod}"'] & \mathrm{Hom}_A(a, A) \arrow[d, "\mathrm{cod}"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \wr^{\circ} a \arrow[r, phantom, "\simeq"] \arrow[dr, "\mathrm{dom}"'] & \mathrm{Hom}_A(A, a) \arrow[d, "\mathrm{dom}"] \\ & A \end{tikzcd}$$

$\rightsquigarrow$ $A \wr A$ and $A \wr^{\circ} A$ are twisted versions of A^{2}.

Theorem. The modules

$$\begin{tikzcd} & A \downarrow A \arrow[dl] \arrow[dr, two heads] & \\ A \times A^{\circ} & & 1 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & A \downarrow^{\circ} A \arrow[dl] \arrow[dr, two heads] & \\ 1 & & A^{\circ} \times A \end{tikzcd}$$

are *duals* in the monoidal bicategory of modules:

$$\left(\begin{tikzcd} & A \times A \downarrow^{\circ} A \arrow[dl] \arrow[dr, two heads] & \\ A & & A \times A^{\circ} \times A \end{tikzcd} \right) \underset{A \times A^{\circ} \times A}{\otimes} \left(\begin{tikzcd} & A \downarrow A \times A \arrow[dl] \arrow[dr, two heads] & \\ A \times A^{\circ} \times A & & A \end{tikzcd} \right)$$

$$\cong$$

$$\left(\begin{tikzcd} & A^{2} \arrow[dl, "\mathrm{cod}"'] \arrow[dr, two heads, "\mathrm{dom}"] & \\ A & & A \end{tikzcd} \right)$$

as modules from A to A.

Yoneda Lemma $\rightsquigarrow$ A^{2} is the unit for $\otimes_A$.

Theorem. The 2-category $\mathcal{CAT}_\infty$ has a duality involution

$$
\begin{tikzcd}
 & \left\{\begin{matrix}\text{modules from}\\ A\times B^{\circ}\times B \text{ to } B\times C\end{matrix}\right\} \arrow[dr, "A\times B \nrightarrow B^{\circ}B \,\otimes_{A\times B^{\circ}\times B}\, -"] & \\
\left\{\begin{matrix}\text{modules from}\\ A\times B^{\circ} \text{ to } C\end{matrix}\right\} \arrow[ur, "-\times B"] \arrow[rr, dashed, bend left, "\simeq"] & & \left\{\begin{matrix}\text{modules from}\\ A \text{ to } B\times C\end{matrix}\right\} \arrow[ll, dashed, bend left, "\simeq"] \arrow[dl, "-\times B^{\circ}"] \\
 & \left\{\begin{matrix}\text{modules from}\\ A\times B^{\circ} \text{ to } B\times B^{\circ}\times C\end{matrix}\right\} \arrow[ul, "- \,\otimes_{B\times B^{\circ}\times C}\, B \nrightarrow B\times C"] &
\end{tikzcd}
$$

Part 2: a classifying left fibration for CAT_∞

defn (Weber). A left fibration $S_* \overset{u}{\twoheadrightarrow} S$ is classifying when the functor $\text{fun}(A, S) \longrightarrow \text{leftfib}/_A$ given by

$$\begin{tikzcd} \int f \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & S_* \arrow[d, two heads, "u"] \\ A \arrow[r, "f"'] & S \end{tikzcd}$$

is fully faithful for all A.

IDEA: Take $S \in CAT_\infty$ to be the ∞-category of U-small ∞-groupoids and take S_* to be the ∞-category of pointed ∞-groupoids.

Again we work with the quasi-categorical model of ∞-categories.

Strategies for constructing the classifying left fibration $\mathcal{S}_* \xrightarrow{u} \mathcal{S}$

- "unstraightening id": define

  $\mathcal{S}$ = the homotopy coherent nerve of a cartesian closed category of spaces

  $\mathcal{S}_*$ = the slice quasi-category $^{*/}\mathcal{S}$

- via locality of left fibrations: take

  $\{\Delta[n] \to \mathcal{S}\} \simeq$ "{U-small left fibrations over $\Delta[n]$}"

  $\{\Delta[n] \to \mathcal{S}_*\} \simeq$ "{U-small left fibrations over $\Delta[n]$ together with a global section}"

- model-independently: define

  $\mathcal{S}$ = the free colimit completion of the one-point space $*$

  $\mathcal{S}_* = \mathrm{Hom}_{\mathcal{S}}(*, \mathcal{S})$

  $\leadsto$ easy to verify $\mathcal{S}_* \xrightarrow{u} \mathcal{S}$ is a left fibration

To define a homotopy coherent functor $\mathrm{Fun}(A,\mathcal{S}) \xrightarrow{\int} \mathrm{LFib}/A$ use the microcosm principle:

Prop. $\mathrm{coCart}(\mathcal{K}) \xrightarrow{\mathrm{cod}} \mathcal{K}$ is a cartesian fibration of $(\infty,2)$-categories:

- locally a cocartesian fibration of ∞-categories, and whiskering defines a cartesian functor
- globally a cartesian fibration up to homotopy

Cor. Cocartesian cocones lift through $\mathrm{coCart}(\mathcal{K}) \xrightarrow{\mathrm{cod}} \mathcal{K}$

The functor $\mathrm{Fun}(A,\mathcal{S}) \xrightarrow{\int} \mathrm{LFib}/A$ is defined by one such lift.

The proof that $\mathrm{Fun}(A,\mathcal{S}) \xrightarrow{\int} \mathrm{LFib}/A$ is fully faithful is a long story that we won't get into here.

Summary:

The 2-category of ∞-categories $\mathcal{CAT}_\infty$ is a weak 2-topos:

(0) a cartesian closed 2-category that has

(1) weak finite limits,

(γ) a duality involution, and

(2) a classifying left fibration.

References:

- Mark Weber "Yoneda structures from 2-toposes"
  Applied Categorical Structures 2007
- Emily Riehl + Dominic Verity "Elements of ∞-category theory"
  draft available at www.math.jhu.edu/~eriehl/elements.pdf

Why might we care that $\mathcal{CAT}_\infty$ is a 2-topos?

For $A \in \mathcal{CAT}_\infty$ define $PA := S^{A^\circ}$.

Declare $A \overset{f}{\to} B \in \mathcal{CAT}_\infty$ to be admissible if the module $\mathrm{Hom}_B(f, B)$ is classified by some $B \times A^\circ \longrightarrow S$.

Note $A \in \mathcal{CAT}_\infty$ is admissible just when A^2 is classified by a functor $A \times A^\circ \longrightarrow S$ which transposes to define $A \xrightarrow{\textrm{よ}} PA$.

Theorem (Weber). Any 2-topos specifies a good Yoneda structure.

Thank you!