Internal languages of higher toposes

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The theorem

Theorem (S.)

Every Grothendieck $(\infty,1)$-topos can be presented by a model category that interprets homotopy type theory with:

- Σ*-types, a unit type,* Π*-types with function extensionality, and identity types.*
- *Strict universes, closed under the above type formers,* $\leftarrow$ *new! and satisfying univalence and the propositional resizing axiom.*

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- What do all these words mean?
- Why should I care?

Outline

1. What is internal logic?
2. What are higher toposes?
3. What is homotopy type theory?
4. The theorem: the idea
5. The theorem: the proof

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Toposes

Definition

A Grothendieck topos is a left-exact-reflective subcategory of a presheaf category, or equivalently the category of sheaves on a site.

It shares many properties of $\mathcal{S}et$, such as:

- finite limits and colimits.
- disjoint coproducts and effective equivalence relations.
- locally cartesian closed.
- a subobject classifier $\Omega = \{\bot, \top\}$.

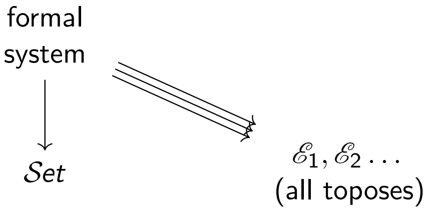
An elementary topos is any category with these properties.

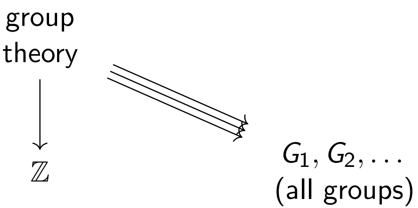
Basic principle

Since most mathematics can be expressed using sets, it can be done internally to any sufficiently set-like category, such as a topos.

Internal logic

Translating into "arrow-theoretic language" by hand is tedious and obfuscating. The internal logic automatically "compiles" a set-like language into objects and morphisms in any topos.





From set theory to type theory

Given two sets X, Y, in ordinary ZF-like set theory we can ask whether $X \subseteq Y$. But this question is meaningless to the category $\mathcal{S}et$; we can only ask about *injections* $X \hookrightarrow Y$. Thus we use a type theory, where each element belongs to only one* type.

$$\begin{array}{ccc} \text{sets} & \rightsquigarrow & \text{types} \\ x \in X & \rightsquigarrow & x : X \end{array}$$

Syntax	Interpretation in a topos $\mathscr{E}$
Type A	Object A of $\mathscr{E}$
Product type $A \times B$	Cartesian product $A \times B$ in $\mathscr{E}$
Term $f(x, g(y)) : C$ using formal variables $x : A$, $y : D$	Composite morphism $A \times D \xrightarrow{1 \times g} A \times B \xrightarrow{f} C$
Dependent type $B(x)$ using a variable $x : A$	Object $B \to A$ of $\mathcal{E}/A$

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Internalizing mathematics

- Ordinary mathematics can nearly always be formalized in type theory, and thereby internalized in any topos.
- This includes definitions, theorems, and also proofs, as long as they use intuitionistic logic.
- Type-theoretic formalization can also be verified by a computer proof assistant.

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Higher toposes

Kind of topos	Objects behave like	Prototypical example
1-topos	sets	$\mathcal{S}et$
2-topos	categories	$\mathcal{C}at$
$(\infty, 2)$-topos	$(\infty, 1)$-categories	$(\infty, 1)$-$\mathcal{C}at$
$(2, 1)$-topos	groupoids	$\mathcal{G}pd$
$(\infty, 1)$-topos	∞-groupoids (spaces)	∞-$\mathcal{G}pd$
$(n, 1)$-topos	$(n-1)$-groupoids	$(n-1)$-$\mathcal{G}pd$

2-toposes and $(\infty, 2)$-toposes are extra hard because:

1. They are not *locally* cartesian closed.
2. $(-)^{\mathsf{op}}$ is hard to deal with and hard to do without.

Today: $(n, 1)$-toposes for $2 \leq n \leq \infty$.
Think $n = \infty$ or $n = 2$, as you prefer.

$(n, 1)$-toposes

Definition (Toen–Vezossi, Rezk, Lurie)

A Grothendieck $(n, 1)$-topos, for $1 \le n \le \infty$, is an accessible* left-exact-reflective subcategory of a presheaf $(n, 1)$-category, or equivalently the category of $(n, 1)$-sheaves on an $(n, 1)$-site*.

It shares many properties of the $(n, 1)$-category of $(n-1)$-groupoids:

- finite limits and colimits.
- disjoint coproducts
- effective quotients of n-efficient groupoids.
- locally cartesian closed.
- a subobject classifier Ω.
- classifiers for small $(n-2)$-truncated morphisms.

(An *elementary* $(n, 1)$-topos should have some of the same properties. But that definition is still negotiable; we have essentially no examples yet.)

Example #1: promoted 1-toposes

Example

Any 1-site $(\mathbb{C}, J)$ is also an $(n, 1)$-site, and any Grothendieck 1-topos $\mathcal{S}h_1(\mathbb{C}, J)$ is the 0-truncated objects in an $(n, 1)$-topos $\mathcal{S}h_n(\mathbb{C}, J)$.

Extends the “set theory” of $\mathcal{S}h_1(\mathbb{C}, J)$ with higher category theory.

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Example

$\mathcal{E}$ a small 1-topos, J its coherent top. $\Rightarrow$ $\mathcal{S}h_2(\mathcal{E}, J)$ a $(2, 1)$-topos.

1. Internal category theory in $\mathcal{S}h_2(\mathcal{E}, J)$ includes indexed category theory over $\mathcal{E}$, but phrased just like ordinary category theory; no need to manually manage indexed families.
2. The internal logic of $\mathcal{S}h_2(\mathcal{E}, J)$ includes the stack semantics of $\mathcal{E}$, expanding its internal logic to unbounded quantifiers (e.g. "there exists an object").

This isn't the topos you're looking for

Warning

$\mathcal{S}h_n(\mathbb{C}, J)$ is not, in general, equivalent to the $(n, 1)$-category of internal $(n-1)$-groupoids in $\mathcal{S}h_1(\mathbb{C}, J)$.

1. The former allows pseudonatural morphisms (inverts weak equivalences).
2. When $n = \infty$, the latter is "hypercomplete" but the former may not be.
3. The 0-truncated objects in the latter don't even recover $\mathcal{S}h_1(\mathbb{C}, J)$, but its exact completion.

Example #2: higher group actions

A monoid acts on sets; a monoidal groupoid acts on groupoids.

Example

The one-object groupoid $\mathcal{B}\mathbb{Z}$ associated to the abelian group $\mathbb{Z}$ is monoidal. A $\mathcal{B}\mathbb{Z}$-action on a groupoid $\mathcal{G}$ consists of, for each $x \in \mathcal{G}$, an automorphism $\phi_x : x \xrightarrow{\sim} x$, such that for all $\psi : x \xrightarrow{\sim} y$ in $\mathcal{G}$ we have $\psi \circ \phi_x = \phi_y \circ \psi$.

Note that $\mathcal{B}\mathbb{Z}$ cannot act nontrivially on a set; we need the (2,1)-topos $\mathcal{B}\mathbb{Z}$-$\mathcal{G}pd$.

Example #3: orbifolds

Definition

An orbifold is a space that "looks locally" like the quotient of a manifold by a group action.

Example

When $\mathbb{Z}/2$ acts on $\mathbb{R}^2$ by 180° rotation, the quotient is a cone, with $\mathbb{Z}/2$ "isotropy" at the origin.

Where does this "quotient" take place?

- The 1-category $\mathcal{M}fd$ doesn't have such colimits.
- $\mathcal{S}h_1(\mathcal{M}fd)$ does, but they forget the isotropy groups.
- Sometimes use quotients in the (2,1)-topos $\mathcal{S}h_2(\mathcal{M}fd)$.
- Sometimes need $\mathcal{S}h_2(\mathcal{O}rb)$, with $\mathcal{O}rb$ a (2,1)-category of smooth groupoids.

Example #4: parametrized spectra

A spectrum is, to first approximation, an ∞-groupoid analogue of an abelian group.

Example

The category of ∞-groupoid-indexed families of spectra is an $(\infty,1)$-topos.

This is some special ∞-magic: set-indexed families of abelian groups are not a 1-topos!

"Higher-order" versions of this are used for Goodwillie calculus.

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Equality and identity

In the internal logic of a 1-topos:

- Equality is a proposition $\mathrm{Eq}_A(x, y)$ depending on $x : A$ and $y : A$, i.e. a relation $\mathrm{Eq}_A : A \times A \to \Omega$.
- Semantically, the diagonal $A \to A \times A$, which is a subobject.

In a higher topos:

- The diagonal $A \to A \times A$ is no longer monic.
- But we can regard it as a family of types: the identity type $\mathrm{Id}_A(x, y)$ depending on $x : A$ and $y : A$.
- We call the elements of $\mathrm{Id}_A(x, y)$ identifications of x and y. Can think of them as isomorphisms in a groupoid.
- Everything we can say inside of type theory can be automatically transported across any identification.

Object classifiers

Definition

An object classifier in $\mathcal{E}$ is a map $\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$ such that pullback $\mathcal{E}(A, \mathcal{U}) \longrightarrow (\mathcal{E}/A)^{\text{core}}$ is fully faithful: any pullback of it is a pullback in a unique way.

Examples

1. A 1-topos has a classifier $\top : 1 \to \Omega$ for all subobjects.
2. An $(\infty, 1)$-topos has classifiers for all κ-small morphisms, for arbitrarily large regular cardinals κ.
3. An $(n, 1)$-topos has classifiers for κ-small $(n-2)$-truncated morphisms (e.g. $\mathcal{S}et_*{}^{\text{core}} \to \mathcal{S}et^{\text{core}}$ in $\mathcal{G}pd$).

Univalence

In type theory, an object classifier becomes a universe type $\mathcal{U}$, whose elements are types. The full-faithfulness of $\mathcal{E}(A,\mathcal{U}) \longrightarrow (\mathcal{E}/A)^{\text{core}}$ becomes Voevodsky's univalence axiom:

Univalence Axiom

For $X : \mathcal{U}$ and $Y : \mathcal{U}$, the identity type $\mathsf{Id}_{\mathcal{U}}(X, Y)$ is canonically equivalent* to the type of equivalences $X \simeq Y$.

Since anything can be transported across identifications, this implies that equivalent types are indistinguishable.

Homotopy type theory

Homotopy Type Theory (HoTT)

The study of type theories inspired by this interpretation, generally including univalence and other enhancements such as higher inductive types.

For example:

- Book HoTT is Martin-Löf Type Theory with axioms for univalence and higher inductive types.
- Cubical type theories are computationally adequate, with rules instead of axioms.

However, no cubical type theories are yet known to have general $(\infty,1)$-topos-theoretic semantics. Today we stick to Book HoTT.

Applications of HoTT as an internal language

1. All of ordinary (constructive) mathematics can be internalized in all higher toposes.
2. Prove theorems from homotopy theory using new techniques of type theory, and deduce that they are true in all higher toposes. (E.g. HFLL, ABFJ: Blakers–Massey theorems)
3. Augment HoTT with synthetic axioms or modalities to work with special classes of higher toposes.
4. Work in higher toposes without needing simplicial sets — fully rigorous and computer-formalizable.

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Coherence and strict equality

Problem

A higher topos is a weak higher category, with universal properties up to equivalence. But operations in type theory obey laws up to definitional equality.

What's that?

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A higher topos is a weak higher category, with universal properties up to equivalence. But operations in type theory obey laws up to definitional equality.

What's that?

There are (at least) two "senses in which" elements x and y of a type A can be "the same".

1. The identity type $\mathsf{Id}_A(x, y)$, whose elements are identifications (paths, homotopies, isomorphisms, equivalences). There can be more than one identification between two elements, and transporting along them can be nontrivial.
2. The definitional equality $x \equiv y$ obtained by expanding definitions, e.g. if $f(x) :\equiv x^2$ then $f(y+1) \equiv (y+1)^2$. Algorithmic and unique, and transporting carries no info.

An idea that I don't recommend

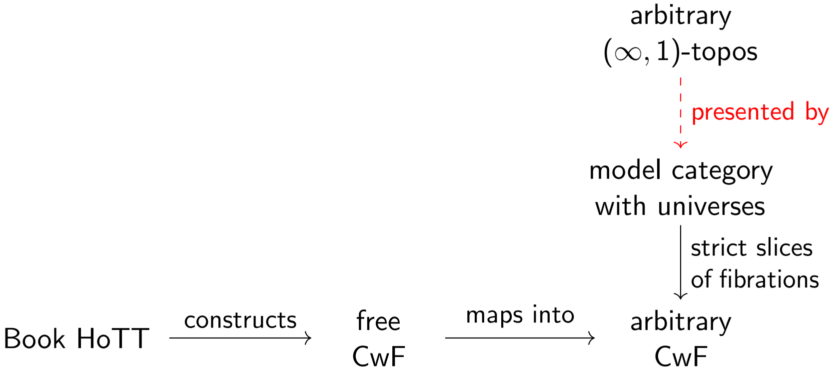
Idea

Weaken type theory to match higher categories, e.g. omit definitional equality.

But strictness is a big part of the advantage of type theory over explicit arrow-theoretic reasoning. Being able to use $1 + 1$ and 2 literally interchangeably is very important for our sanity. This gets even worse in a higher category where we have not only homotopies but higher coherence homotopies all the time!

We need strict models for actual Grothendieck $(\infty,1)$-toposes, with strict equalities that behave like definitional equalities.

From univalent universes to $(\infty, 1)$-toposes



From pseudo to strict

In the (2,1)-topos $[\mathbb{D}^{\mathrm{op}}, \mathcal{G}pd]$, every pseudofunctor $X : \mathbb{D}^{\mathrm{op}} \to \mathcal{G}pd$ is equivalent to a strict one. Not every pseudonatural transformation $X \rightsquigarrow Y$ is equivalent to a strict $X \to Y$, but:

Lemma

For any $Y \in [\mathbb{D}^{\mathrm{op}}, \mathcal{G}pd]$ there is a strict $\mathsf{C}^{\mathbb{D}} Y$ and a bijection between pseudonatural $X \rightsquigarrow Y$ and strict $X \to \mathsf{C}^{\mathbb{D}} Y$.

Proof.

A pseudonatural $f : X \rightsquigarrow Y$ assigns to each $x \in X(c)$

- An image $f_x(x) \in Y(c)$, but also
- An isomorphism $\gamma^*(f_x(x)) \cong f_{x'}(\gamma^*(x))$ for all $\gamma : x' \to x$ in $\mathbb{D}$,
- Satisfying a coherence condition.

Thus, we define $\mathsf{C}^{\mathbb{D}} Y(c)$ to consist of all these data. □

Coflexible objects

Definition (Blackwell-Kelly-Power)

Y is coflexible if the canonical map $Y \to \mathsf{C}^{\mathbb{D}} Y$ has a strict retraction.

Lemma

If Y is coflexible, then every pseudonatural transformation $X \rightsquigarrow Y$ is isomorphic to a strict one $X \to \mathsf{C}^{\mathbb{D}} Y \to Y$.

Idea

Interpret types as coflexible objects.

- Get a well-behaved 1-category of strict morphisms.
- Can still capture all the “pseudo information”.

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Overview

Theorem

Every Grothendieck $(\infty,1)$-topos can be presented by a model category that interprets homotopy type theory with strict univalent universes.

1. Any $(\infty,1)$-topos is a left exact localization of a presheaf one.
2. A Quillen model category of injective simplicial presheaves presents an $(\infty,1)$-presheaf topos, and models all of type theory except universes.
3. Use coflexibility to characterize the injective fibrations and build a universe for presheaves.
4. Localize internally to build a universe of sheaves.

Type-theoretic model categories

A Quillen model category $\mathcal{E}$ is a 1-category with structure to present an $(\infty,1)$-category, including (co)fibrations and weak equivalences.

If $\mathcal{E}$ is locally cartesian closed, right proper, and its cofibrations are the monomorphisms, then we can interpret "types in context Γ" as fibrations in $\mathcal{E}/\Gamma$ to model a type theory with:

- a unit type and Σ-types (fibrations contain the identities and are closed under composition).
- Identity types (as path objects — Awodey–Warren, etc.).
- Π-types satisfying function extensionality (dependent products preserve fibrations).

What about universes?

- In type theory, we want universes that are *closed under* all the other rules.
- If κ is inaccessible, the κ-small morphisms are closed under everything.
- But, the classifier of κ-small morphisms in an $(\infty,1)$-topos only classifies them up to equivalence!
- We need a fibration $\pi : \widetilde{\mathcal{U}} \to U$ in a model category that classifies κ-small fibrations by 1-categorical pullback.

Universes in presheaves

Definition

If $\mathcal{E} = [\mathbb{C}^{\mathsf{op}}, \mathcal{S}et]$ is a presheaf category, define a presheaf $\mathcal{U}$ where

$$\mathcal{U}(c) = \Big\{\kappa\text{-small fibrations over } よc = \mathbb{C}(-, c)\Big\}.$$

Functorial action is by pullback.

This takes a bit of work to make precise:

- $\mathcal{U}(c)$ must be a *set* containing at least one representative for each *isomorphism class* of such κ-small fibrations.
- Chosen cleverly to make pullback *strictly* functorial.

Universes in presheaves, II

Similarly, we can define $\widetilde{\mathcal{U}}$ to consist of κ-small fibrations equipped with a section. We have a κ-small projection $\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$.

Theorem

Every κ-small fibration is a pullback of π.

But π may not itself be a fibration! All we can say is that its pullback along any map $x : よc \to \mathcal{U}$, with $よc$ representable, is a fibration (namely the fibration that "is" $x \in U(c)$).

It works if the generating acyclic cofibrations have representable codomain (e.g. Voevodsky's simplicial set model), but in general we can't assume that.

Injective model structures

$\mathcal{S}$ = simplicial sets, $\mathbb{D}$ = a small simplicially enriched category.

Theorem

The category $[\mathbb{D}^{\mathrm{op}}, \mathcal{S}]$ of simplicially enriched presheaves has an injective model structure such that:

1. *The weak equivalences are pointwise.*
2. *The cofibrations are pointwise, hence are the monomorphisms in $[\mathbb{D}^{\mathrm{op}}, \mathcal{S}]$.*
3. *It is locally cartesian closed and right proper.*
4. *It presents the (∞,1)-category of (∞,1)-presheaves on the small (∞,1)-category presented by $\mathbb{D}$.*

So it models everything but universes.

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- *The fibrations are ...?????*

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Understanding injective fibrancy

When is $X \in [\mathbb{D}^{\mathrm{op}}, \mathcal{S}]$ injectively fibrant? We want to lift in

$$\begin{array}{ccc} A & \xrightarrow{g} & X \\ {\scriptstyle i}\big\downarrow{\scriptstyle \sim} & \nearrow & \\ B & & \end{array}$$

where $i : A \to B$ is a pointwise acyclic cofibration.
If X is pointwise fibrant, then for all $d \in \mathbb{D}$ we have a lift

$$\begin{array}{ccc} A_d & \xrightarrow{g_d} & X_d \\ {\scriptstyle i_d}\big\downarrow{\scriptstyle \sim} & \nearrow{\scriptstyle h_d} & \\ B_d & & \end{array}$$

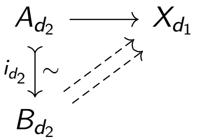
but these may not fit together into a natural transformation $B \to X$.

Naturality up to homotopy

Naturality would mean that for any $\delta : d_1 \to d_2$ in $\mathbb{D}$ we have $X_\delta \circ h_{d_2} = h_{d_1} \circ B_\delta$. This may not hold, but we do have

$$X_\delta \circ h_{d_2} \circ i_{d_2} = X_\delta \circ g_{d_2} = g_{d_1} \circ A_\delta = h_{d_1} \circ i_{d_1} \circ A_\delta = h_{d_1} \circ B_\delta \circ i_{d_2}.$$

Thus, $X_\delta \circ h_{d_2}$ and $h_{d_1} \circ B_\delta$ are both lifts in the following:

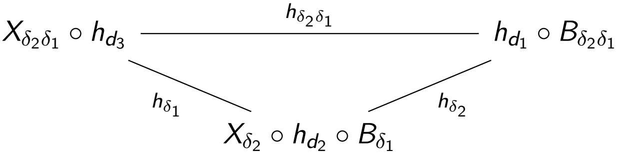


Since lifts between acyclic cofibrations and fibrations are unique up to homotopy, we do have a homotopy

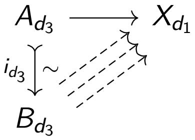
$$h_\delta : X_\delta \circ h_{d_2} \sim h_{d_1} \circ B_\delta.$$

Coherent naturality

Similarly, given $d_1 \xrightarrow{\delta_1} d_2 \xrightarrow{\delta_2} d_3$, we have a triangle of homotopies



whose vertices are lifts in the following:



Thus, homotopy uniqueness of lifts gives us a 2-simplex filler.

The coherent morphism coclassifier

Conclusion

If X is pointwise fibrant, then any lifting problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ \sim\big\downarrow & \nearrow & \\ B & & \end{array}$$

is "solved" by some homotopy coherent natural transformation.

For X to be injectively fibrant, need to be able to replace this by a strict natural transformation.

Coflexibility again

Fact

For any $X \in [\mathbb{D}^{\mathsf{op}}, \mathcal{S}]$ there is a cobar construction $\mathsf{C}^{\mathbb{D}}(Y)$ and a bijection between homotopy coherent transformations $X \rightsquigarrow Y$ and strict ones $X \to \mathsf{C}^{\mathbb{D}}(Y)$.

Definition

X is coflexible if the canonical map $X \to \mathsf{C}^{\mathbb{D}}X$ has a strict retraction.

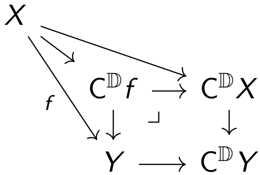
In this case, any homotopy coherent transformation $B \rightsquigarrow X$ is homotopic to a strict one $B \to \mathsf{C}^{\mathbb{D}}X \to X$.

Injective fibrations

Theorem

$X \in [\mathbb{D}^{\mathrm{op}}, \mathcal{S}]$ is injectively fibrant if and only if it is pointwise fibrant and coflexible.

More generally, any $f : X \to Y$ can be factored by pullback:



Theorem

$f : X \to Y$ is an injective fibration if and only if it is a pointwise fibration and the map $X \to \mathrm{C}^{\mathbb{D}} f$ has a retraction over Y.

Semi-algebraic fibrations

Definition

A semi-algebraic injective fibration is a map $f : X \to Y$ with

1. The property of being a pointwise fibration, and
2. The structure of a retraction for $X \to \mathrm{C}^{\mathbb{D}} f$.

Now define $\mathcal{U} \in [\mathbb{D}^{\mathrm{op}}, \mathcal{S}]$ (and similarly $\widetilde{\mathcal{U}}$ and $\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$):

$$\mathcal{U}(d) = \left\{ \kappa\text{-small semi-algebraic injective fibrations over } \text{よ}d \right\}.$$

Theorem

$\pi : \widetilde{\mathcal{U}} \to \mathcal{U}$ is a (semi-algebraic) injective fibration.

Proof.

Glue together the semi-algebraic structures over each $\text{よ}d$. □

Sheaf universes

Given a left exact localization $L_S[\mathbb{D}^{op}, \mathcal{S}]$:

1. Using a technical result of Anel–Biedermann–Finster–Joyal (2019, forthcoming), we can ensure that left exactness of S-localization is pullback-stable.
2. Then for any $f : X \twoheadrightarrow Y$ we can construct *in the internal type theory of* $[\mathbb{D}^{op}, \mathcal{S}]$ a fibration $\mathsf{isLocal}_S(f) \twoheadrightarrow Y$.
3. Define a semi-algebraic local fibration to be a semi-algebraic injective fibration equipped with a section of $\mathsf{isLocal}_S(f)$.
4. Now use the same approach.

The theorem, again

Theorem (S.)

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- *Σ-types, a unit type, Π-types with function extensionality, and identity types.*
- *Strict universes, closed under the above type formers, and satisfying univalence and the propositional resizing axiom.*

What's next?

- These model categories have higher inductive types too; are the universes closed under them?
- Can we construct any non-Grothendieck higher toposes?
- Can cubical type theories also be interpreted in higher toposes?