A Solution for the Compositionality Problem of Dinatural Transformations

Guy McCusker∗  Alessio Santamaria†

Dinatural transformations are a generalisation of the well-known natural transformations, as such they are ubiquitous in Mathematics and Computer Science. They appeared for the first time in [18] in the context of Algebraic Topology, where the notion of (co)end of a functor was introduced; lately they were formally defined in [3]. Given functors $F, G: \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{D}$, a dinatural transformation $\varphi: F \to G$ is a family $\{\varphi_A: F(A, A) \to G(A, A)\}_{A \in \mathbb{C}}$ such that for all $f: A \to B$ in $\mathbb{C}$ the following hexagon commutes:

\[
\begin{array}{ccc}
F(A, A) & \xrightarrow{\varphi_A} & G(A, A) \\
\downarrow F(f, 1) & & \downarrow G(1, f) \\
F(B, A) & & G(A, B) \\
\downarrow F(1, f) & & \downarrow G(f, 1) \\
F(B, B) & \xrightarrow{\varphi_B} & G(B, B)
\end{array}
\]

A classical example is the family of evaluation maps $(eval_{A, B} : A \times (A \Rightarrow B) \to B)$ in any cartesian closed category $\mathbb{C}$: the transformation $eval$ is natural in $B$ and dinatural in $A$.

Dinatural transformations, however, suffer from a troublesome shortcoming: they do not compose. This remarkable problem was already known to their discoverers: many studies have been conducted about them [1, 2, 5, 6, 9, 11, 12, 14, 15, 16, 17], and many attempts have been made to find a proper calculus for dinatural transformations, but only ad hoc solutions have been found and, ultimately, they have remained poorly understood. We present a sufficient and essentially necessary condition for two arbitrary, consecutive dinatural transformations $\varphi$ and $\psi$ for the composite $\psi \circ \varphi$ to be dinatural, thus solving the compositionality problem of dinatural transformations in its full generality [10]. We were inspired by the work of Eilenberg and Kelly on extranatural transformations [4], which are less general than dinaturals and also fail to compose: the authors associated to each extranatural a graph, the archetype of a string diagram, that captures their naturality properties. We extended such graphical calculus to dinatural transformations; for example, consider the transformation $eval$ as above: its domain is the functor $T: \mathbb{C} \times \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$ where $T(X, Y, Z) = X \times (Y \Rightarrow Z)$, while the codomain is $id_{\mathbb{C}}$. The graph of $eval$ is:

\[
\Gamma(eval) = \begin{array}{c}
\mathbb{C} \\
\mathbb{C}^{op} \\
\mathbb{C}
\end{array}
\]

The three upper boxes correspond to the arguments of $T$, while the lower one to $id_{\mathbb{C}}$. Graphs of consecutive dinatural transformations $\varphi: F \to G$ and $\psi: G \to H$ can be composed by “glueing” them together along the $G$-boxes. Our result asserts that if the composite graph $\Gamma(\psi) \circ \Gamma(\varphi)$ is acyclic, then $\psi \circ \varphi$ is indeed dinatural. The proof exploits the theory of Petri Nets [13], of which these graphs are a particular example, by translating the dinaturality property of $\psi \circ \varphi$ into a reachability problem for the Petri Net $\Gamma(\psi) \circ \Gamma(\varphi)$. We can now finally define a generalised functor category $(\mathbb{C}, \mathbb{D})$ of mixed-variance functors and (partially) dinatural transformations; this is the first step towards the formalisation of a generalised Godement calculus as sought by Kelly in [7] in order to describe coherence problems abstractly [8].

∗University of Bath, G.A.McCusker@bath.ac.uk
†University of Bath, A.Santamaria@bath.ac.uk
References


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