The Scott Adjunction

Ivan Di Liberti

CT 2019
7-2019
Plot

The main characters of this talk are:

1. Categorical approaches to model theory;
2. Categorification of the Frm $\leftrightarrow$ Top adjunction;
3. The interplay between the previous two points.

Thus, please stay if you are interested in at least one of the topics.

Structure

1. Logic. Motivation, idea, and some results.
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1. **Logic.** motivation, idea, and some results.
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Categorical model theory is a subfield of categorical logic aiming to describe the relevant categorical properties of the categories of models of some theory. It was extensively developed by Makkai and Paré in their well-known book [80s].

Motto: Categorical model theory ↔ accessible categories

Since then, some hypotheses have very often been added in order to smooth the theory and obtain the same results of the classical model theory:

1. amalgamation property;
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3. a nice enough forgetful functor $U: A \to \text{Set}$;
4. every map is a monomorphism;
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Thm. (Rosicky, Beke, Lieberman)

A category $\mathcal{A}$ is equivalent to an abstract elementary class iff:

1. it is an accessible category with directed colimits;
2. every map is a monomorphism;
3. it has a *structural* functor $U : \mathcal{A} \to \mathcal{B}$, where $\mathcal{B}$ is finitely accessible and $U$ is iso-full, nearly full and preserves directed colimits and monomorphisms.
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Quite not what we were looking for, uh?!
This looks a bit artificial, unnatural and not elegant.

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2. When an accessible category with directed colimits admits such a nice forgetful functor?
The Scott Adjunction (Henry, DL)

There is an $\mathcal{S}$:

\[ \text{Acc}_\omega \leftrightarrow \text{Topoi} : \text{pt}. \]

$\text{Acc}_\omega$ is the 2-category of accessible categories with directed colimits, a 1-cell is a functor preserving directed colimits, 2-cells are invertible natural transformations.

$\text{Topoi}$ is the 2-category of Groethendieck topoi. A 1-cell is a geometric morphism and has the direction of the right adjoint. 2-cells are natural transformations between left adjoints.
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The category of points of a locally decidable topos is an AEC.

Thm. (Henry, DL) The unit \( \eta: A \to \text{ptS} A \) is faithful precisely when \( A \) has a faithful functor into Set preserving directed colimits.

Thm. (Henry) There is an accessible category with directed colimits which cannot be axiomatized by a geometric theory.

This problem was originally proposed by Rosicky in his talk "Towards categorical model theory" at the 2014 category theory conference in Cambridge: Show that the category of uncountable sets and monomorphisms between cannot be obtained as the category of points of a topos. Or give an example of an abstract elementary class that does not arise as the category points of a topos.
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The Scott construction

Let \( \mathcal{A} \) be a 0-cell in \( \text{Acc}_\omega \). \( S(\mathcal{A}) \) is defined as the category \( \text{Acc}_\omega(\mathcal{A}, \text{Set}) \).
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Let $\mathcal{A}$ be a 0-cell in $\text{Acc}_\omega$. $S(\mathcal{A})$ is defined as the category $\text{Acc}_\omega(\mathcal{A}, \text{Set})$. Let $f : \mathcal{A} \to \mathcal{B}$ be a 1-cell in $\text{Acc}_\omega$.

\[
\begin{array}{c}
\text{A} \\
\downarrow^f \\
\text{B}
\end{array}
\quad
\begin{array}{c}
S\mathcal{A} \\
\circlearrowright_{f^* \dashv f_*}
\end{array}
\quad
\begin{array}{c}
\text{SB}
\end{array}
\]

$Sf = (f^* \dashv f_*)$ is defined as follows: $f^*$ is the precomposition functor $f^*(g) = g \circ f$. This is well defined because $f$ preserve directed colimits. $f^*$ preserve all colimits and thus has a right adjoint, that we indicate with $f_*$. Observe that $f^*$ preserve finite limits because finite limits commute with directed colimits in $\text{Set}$. 
$S \dashv pt$ is essentially a schizophrenic 2-adjunction induced by the object $\text{Set}$ that inhabits both the 2-categories.
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In this perspective our adjunction, which in this case is a duality, presents $S(\mathcal{A})$ as a free geometric theory attached to the accessible category $\mathcal{A}$ that is willing to axiomatize $\mathcal{A}$. 
The naive Ivan

Is the Scott adjunction the categorification of the Isbell duality between locales and topological spaces? Not precisely.
**The naive Ivan**

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Not precisely.
The geometric picture

- \( \text{Loc} \)
- \( \text{Top} \)
- \( \text{Pos}_\omega \)

\( \text{Loc} \) is the category of Locales. It is defined to be the opposite category of frames, where objects are frames and morphisms are morphisms of frames.

\( \text{Top} \) is the category of topological spaces and continuous mappings between them.

\( \text{Pos}_\omega \) is the category of posets with directed suprema and functions preserving directed suprema.
The geometric picture

\[ \text{Loc} \]

\[ \text{Top} \]

\[ \text{Pos}_\omega \]

\[ \text{O} \]

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\[ \text{S} \]

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\[ \text{ST} \]

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\[ \text{Loc} \rightarrow \text{Top} \leftarrow \text{Pos}_\omega \]

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Ionads!
Ionads

The 2-category of Ionads was introduced by Garner. An ionad $X = (X, \text{Int})$ is a set $X$ together with a comonad $\text{Int} : \text{Set} X \to \text{Set} X$ preserving finite limits. While topoi are the categorification of locales, Ionads are the categorification of the notion of topological space, to be more precise, $\text{Int}$ categorifies the interior operator of a topological space.

Thm. (Garner) The category of coalgebras for a ionad is indicated with $O(X)$ and is a cocomplete elementary topos. A ionad is bounded if $O(X)$ is a Grothendieck topos. Thus one should look at the functor $O : \text{Bion} \to \text{Topoi}$, as the categorification of the functor that associates to a space its frame of open sets.
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\[ \mathcal{O} : \text{Blon} \to \text{Topoi}, \]

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$\text{Blon} \leftarrow \text{Topoi}$

$\emptyset \quad \text{pt} \quad \mathcal{S}$

$\text{Acc}_\omega$
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Replacing bounded Ionads with large bounded Ionads, there exists a right adjoint for \( O \) and a Scott topology-construction \( ST \) such that \( S = O \circ ST \), in complete analogy to the posetal case.
Thm. (DL)

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